Electrical Engineering Foundation

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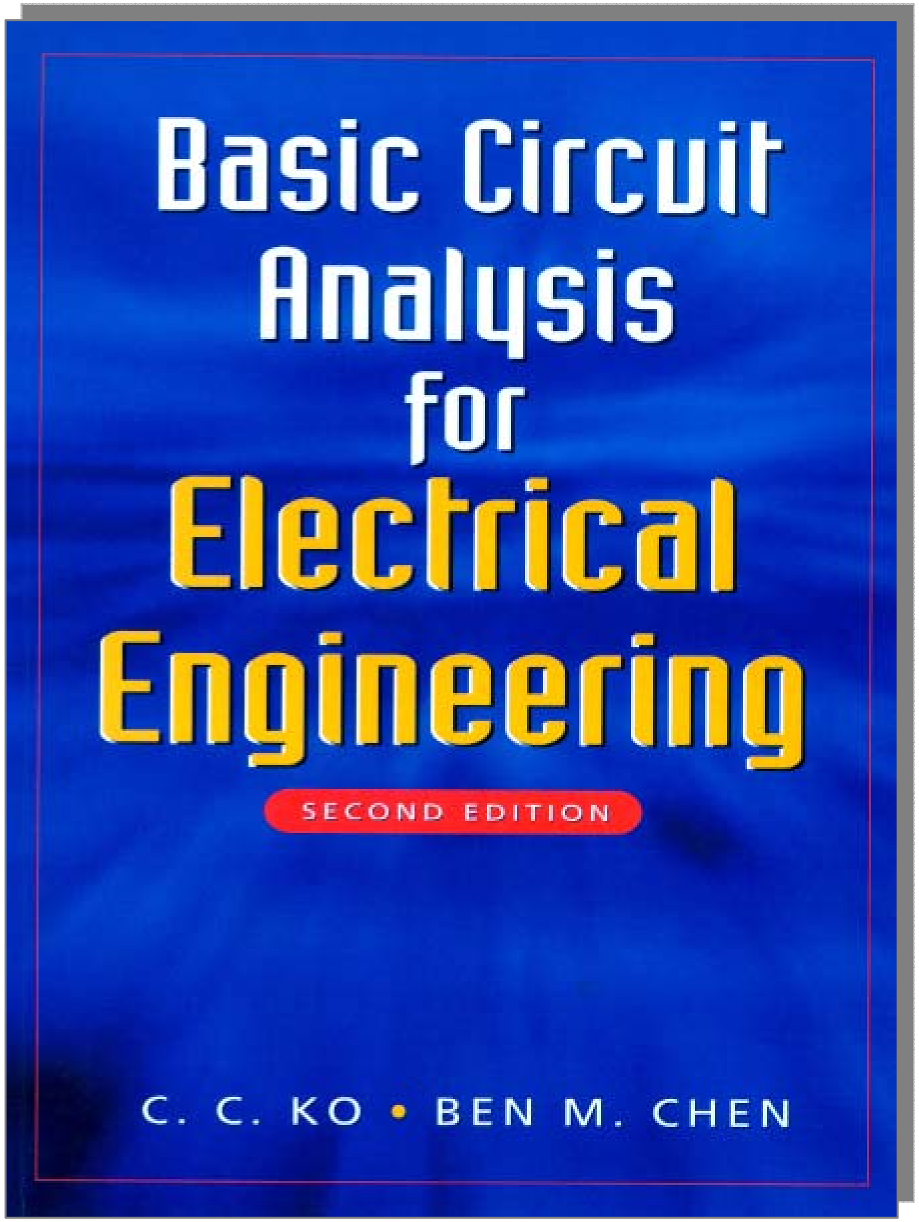
0: Lecture Format & Course Outlines

textbook

or

lecture

notes



Basic Circuit
Analysis
for
Electrical
Engineering

SECOND EDITION

C. C. KO • BEN M. CHEN

Electrical Engineering

Well known electrical engineering companies:

- SingTel, StarHub, M1, ST Engineering, etc…
- Creative Technology (?)
- Chartered Semiconductor (?)

“Yan can Cook”:

- Ingredients + Recipe + (Funny Talk) = Good Food

Electrical Systems:

- Components + Method + (Funny Talk) = Good Electrical System

This Module:

- To introduce basic electrical components & analysis methods.

Food for Thought

There was once a man who was learning archery from a great master.

One day, while practicing, the man's arrow hit the bull eye of the target. He was very excited and ran to tell the master. The master took a look at the arrow and asked him if he knew ***how*** *he had hit the target. "I do not know," said the man.*

The master then told the man he have not mastered archery yet. The man was discouraged but still continued practicing everyday.

Many days later, the man was able to hit the target whenever he shot. He then asked the master to take a look. Again, the master asked the man if he knew ***how*** *the arrow hit the target. The man said, "Yes, I do."*

"Good," said the master, "You know archery now."

It is not enough to hit the target. It is understanding how that is the all-important.

Reference Textbooks

D.E. Johnson, J.R. Johnson and J.L. Hilburn, *Electric Circuits Analysis*, 2nd Ed., Prentice Hall, 1992.

S. A. Boctor, *Electric Circuits Analysis*, 2nd Ed., Prentice Hall, 1992.

Lectures

Lectures will follow closely (but not 100%) the materials in the textbook.

However, certain parts of the textbook will not be covered and examined and this will be made known during the classes.

Attendance is essential.

ASK any question at any time during the lecture.

Tutorials

There are 4 tutorials for this foundation module.

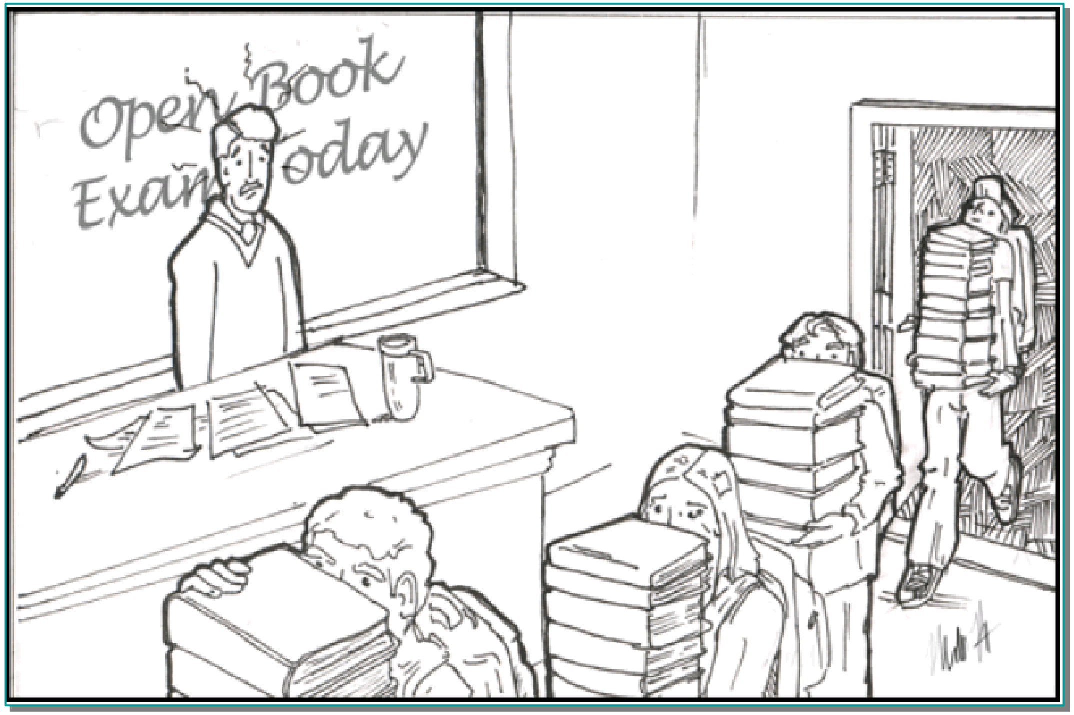
Although you should make an effort to attempt each question before the tutorial, it is NOT necessary to finish all the questions.

Some of the questions are straightforward, but quite a few are difficult and meant to serve as a platform for the introduction of new concepts.

ASK your tutor any question related to the tutorials and the course.

Examination

The examination paper is 2-hour in duration. It will be open book.



Outline of the Course

1. DC Circuit Analysis

SI Units. Voltage, current, power and energy. Voltage and current sources. Resistive circuits. Kirchhoff's voltage and current laws. Nodal and mesh analysis. Ideal and practical sources. Maximum power transfer. Thevenin’s and Norton’s equivalent circuits. Superposition. Dependent sources. Introduction to non-linear circuit analysis.

2. AC Circuits

Root mean square value. Frequency and phase. Phasor. Capacitor and Inductor. Impedance. Power. Power factor. Power factor improvement. Frequency response. Tune circuit. Resonance, bandwidth and Q factor. Periodic signals. Fourier series.

1. SI Units

1.1 Important Quantities and Base SI Units

Length	metre	m
Mass, m	kilogram	kg
Time, t	second	s
Electric current, i	ampere	A
Thermodynamic temperature	kelvin	K
Plane angle	radian	rad

2. DC Circuit Analysis

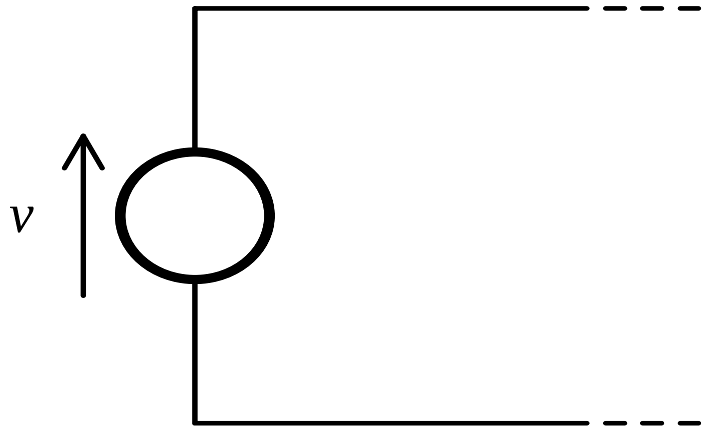
2.1 Voltage Source

Two common dc (direct current) voltage sources are:

- Dry battery (AA, D, C, etc.)
- Lead acid battery in car

Regardless of the load connected and the current drawn, the above sources have the characteristic that the supply voltage will not change very much.

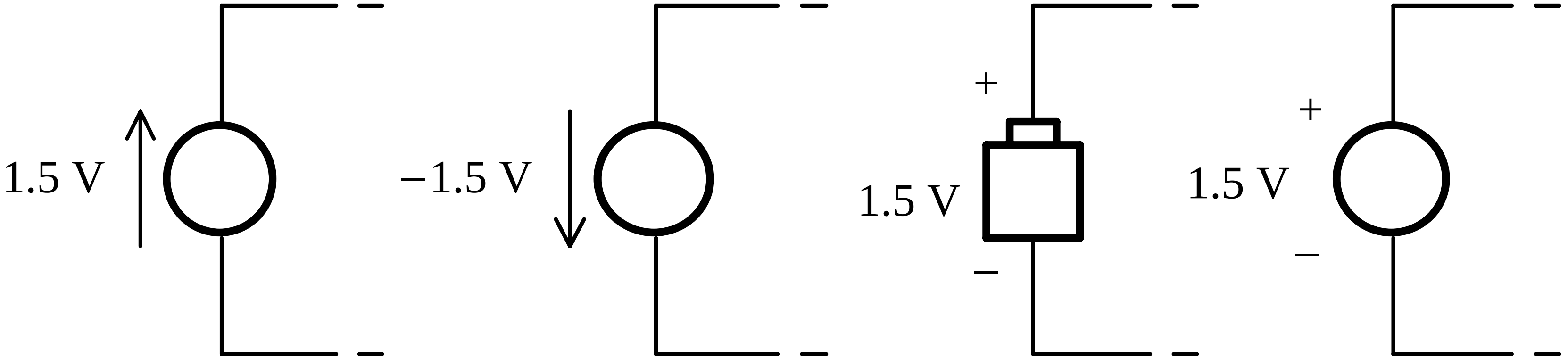
The definition for an ideal voltage source is thus one whose output voltage does not depend on what has been connected to it. The circuit symbol is



Basically, the arrow and the value signifies that the top terminal has a potential of v with respect to the bottom terminal regardless of what has been connected and the current being drawn.

Note that the current being drawn is not defined but depends on the load connected. For example, a battery will give no current if nothing is connected to it, but may be supplying a lot of current if a powerful motor is connected across its terminals. However, in both cases, the terminal voltages will be roughly the same.

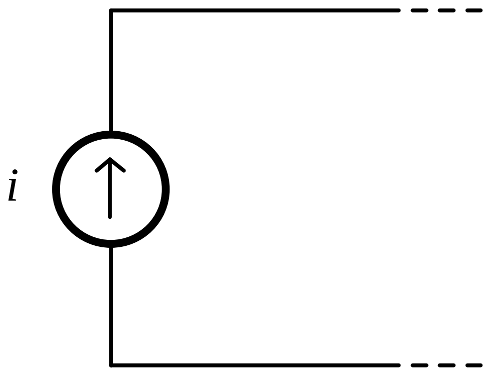
Using the above and other common circuit symbol, the following are identical:



Note that on its own, the arrow does not correspond to the positive terminal. Instead, the positive terminal depends on both the arrow and the sign of the voltage which may be negative.

2.2 Current Source

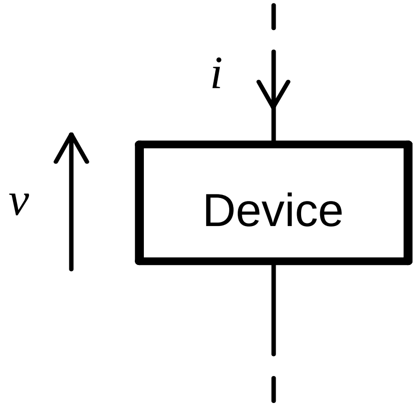
In the same way that the output voltage of an ideal voltage source does not depend on the load and the current drawn, the current delivers by an ideal current source does not depend on what has been connected and the voltage across its terminals. Its circuit symbol is



Note that ideal voltage and current sources are idealisations and do not exist in practice. Many practical electrical sources, however, behave like ideal voltage and current sources.

2.3 Power and Energy

Consider the following device,

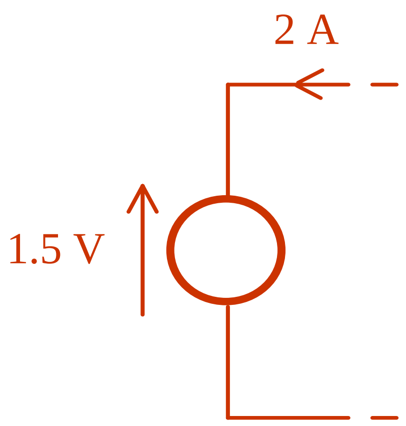


Power Consumed by Device $p = vi$

In 1 second, there are i charges passing through the device. Their electric potential will decrease by v and their electric potential energy will decrease by iv. This energy will have been absorbed or consumed by the device.

The ***power*** or the rate of energy ***consumed*** by the device is thus $p = i\,v$.

Note that $p = v\,i$ gives the power consumed by the device if the voltage and current arrows are opposite to one another. The following examples illustrate this point:

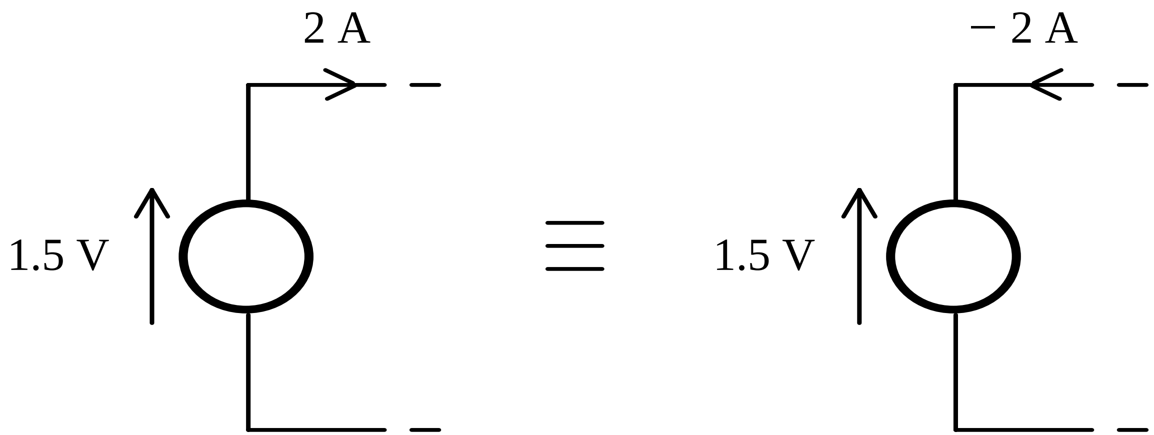


Power consumed/ absorbed by source = 3 W

Energy absorbed in 100 hr = 300 W hr

= 0.3 kW hr

= 0.3 unit in PUB bill

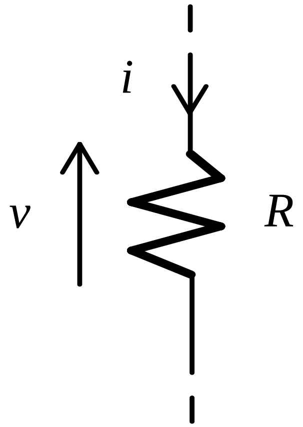


Power supplied by source = 3 W

Power absorbed by source = − 3 W

2.4 Resistor

The symbol for an ideal resistor is



Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship follows Ohm's law:

$$v = iR$$

The power consumed is

$$p = vi = i^2 R = \frac{v^2}{R}$$

Common practical resistors are made of carbon film, wires, etc.

2.5 Relative Power

Powers, voltages and currents are often measured in relative terms with respect to certain convenient reference values. Thus, taking

$$p_{ref} = 1 \text{ mW}$$

as the reference (note that reference could be any value), the power

$$p = 2\text{W}$$

will have a relative value of

$$\frac{p}{p_{ref}} = \frac{2\,\text{W}}{1\text{mW}} = \frac{2\,\text{W}}{10^{-3}\,\text{W}} = 2000$$

The log of this relative power or power ratio is usually taken and given a dimensionless unit of *bel*. The power $p = 2\ W$ is equivalent to

$$\log\left(\frac{p}{p_{ref}}\right) = \log(2000) = \log(1000) + \log(2) = 3.3\,\text{bel}$$

As bel is a large unit, the finer sub-unit, *decibel* or dB (one-tenth of a Bel), is more commonly used. In dB, $p = 2\ W$ is the same as

$$10\log\left(\frac{p}{p_{ref}}\right) = 10\log(2000) = 33\text{dB}$$

As an example:

Reference	**Actual power**	**Relative**	**Power**
p_{ref}	p	p/p_{ref}	$10\log(p/p_{ref})$
1mW	1mW	1	0 dB
1mW	2mW	2	3dB
1mW	10mW	10	10dB
1mW	20mW	$20 = 10 \times 2$	$13\text{dB} = 10\text{dB} + 3\text{dB}$
1mW	100mW	100	20dB
1mW	200mW	$200 = 100 \times 2$	$23\text{dB} = 20\text{dB} + 3\text{dB}$

Although dB measures relative power, it can also be used to measure relative voltage or current which are indirectly related to power.

For instance, taking

$$v_{ref} = 0.1\text{V}$$

as the reference voltage (again reference voltage could be any value), the power consumed by applying v_{ref} to a resistor R will be

$$p_{ref} = \frac{v_{ref}^{\ 2}}{R}$$

Similarly, the voltage

$$v = 1\ \text{V}$$

will lead to a power consumption of $p = \dfrac{v^2}{R}$

The voltage v relative to v_{ref} will then give rise to a relative power of

$$\frac{p}{p_{ref}} = \frac{\frac{v^2}{R}}{\frac{v_{ref}^2}{R}} = \left(\frac{v}{v_{ref}}\right)^2 = \left(\frac{1}{0.1}\right)^2 = 100$$

or in dB:

$$10\log\left(\frac{p}{p_{ref}}\right)\mathrm{dB} = 10\log\left(\frac{v}{v_{ref}}\right)^2\mathrm{dB} = 20\log\left(\frac{v}{v_{ref}}\right)\mathrm{dB} = 20\log\left(\frac{1}{0.1}\right)\mathrm{dB} = 20\mathrm{dB}$$

This is often used as a measure of the relative voltage v/v_{ref} .

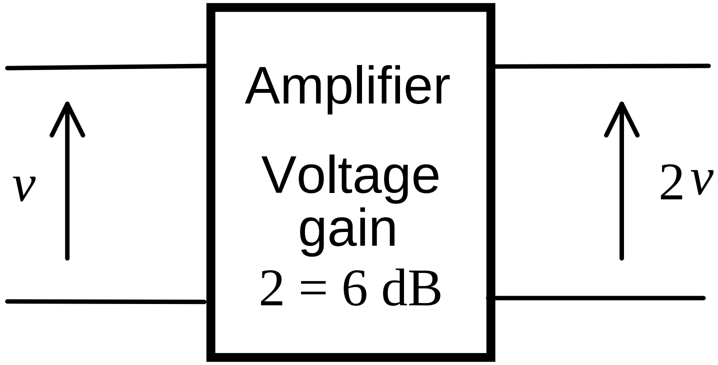
Key point: When you convert relative power to dB, you multiply its log value by 10. You should multiply its log value by 20 if you are converting relative voltage or current.

As an example:

Reference	Actual voltage	Relative	Voltage
v_{ref}	v	v/v_{ref}	$20\log(v/v_{ref})$
0.1 V	0.1 V	1	0dB
0.1 V	$0.1\sqrt{2}$ V	$\sqrt{2}$	3dB
0.1 V	0.2 V	$2 = \sqrt{2} \times \sqrt{2}$	6dB=3dB+3dB
0.1 V	$0.1\sqrt{10}$ V	$\sqrt{10}$	10dB
0.1 V	$0.1\sqrt{20}$ V	$\sqrt{20} = \sqrt{10} \times \sqrt{2}$	13dB=10dB+3dB
0.1 V	1 V	$10 = \sqrt{10} \times \sqrt{10}$	20dB=10dB+10dB

The measure of relative current is the same as that of relative voltage and can be done in dB as well.

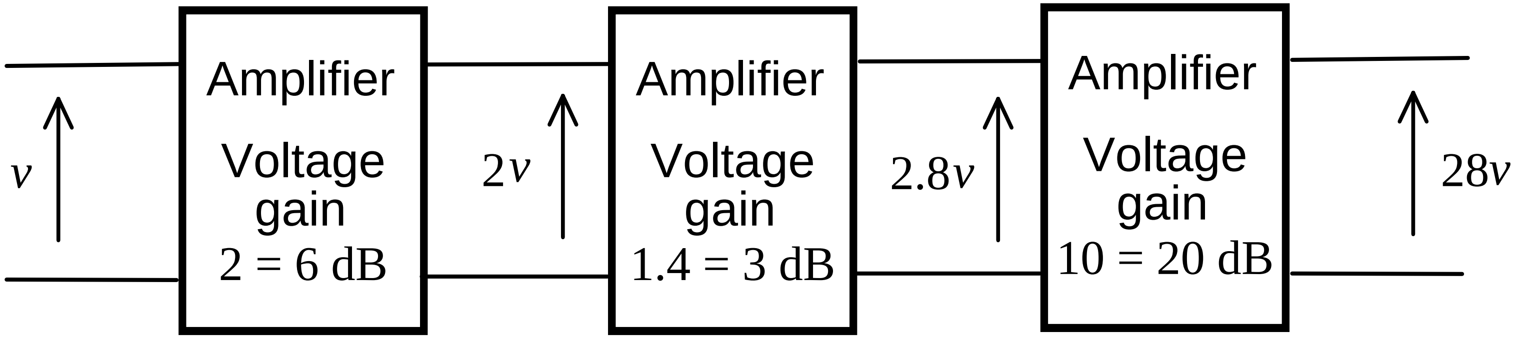
The advantage of measuring relative power, voltage and current in dB can be seen from considering the following voltage amplifier:



The voltage gain of the amplifier is given in terms of the output voltage relative to the input voltage or, more conveniently, in dB:

$$g = \frac{2v}{v} = 2 = 20 \log\left(2\right) \text{dB} = 6 \text{ dB}$$

If we cascade 3 such amplifiers with different voltage gains together:



the overall voltage gain will be

$$g_{total} = 2 \times 1.4 \times 10 = 28$$

However, in dB, it is simply:

$$g_{total} = 6\text{dB} + 3\text{dB} + 20\,\text{dB} = 29\,\text{dB}$$

Under dB which is log based, multiplication becomes addition.

Frequently Asked Questions

Q: Does the arrow associated with a voltage source always point at the + (high potential) terminal?

A: No. The arrow itself is meaningless. As re-iterated in the class, any voltage or current is actually characterized by two things: *its direction and its value*. The arrow of the voltage symbol for a voltage source could point at the – terminal (in this case, the value of the voltage will be negative) or at the + terminal (in this case, its value will be positive).

Q: What is the current of a voltage source?

A: The current of a voltage source is depended on the other part of circuit connected to it.

Frequently Asked Questions

Q: Does a volt source always supply power to other components in a circuit?

A: NO. A voltage source might be consuming power if it is connected to a circuit which has other more powerful sources. Thus, it is a bad idea to pre-determine whether a source is consuming power or supplying power. The best way to determine it is to follow the definition in our text and computer the power. If the value turns out to be positive, then the source will be consuming power. Otherwise, it is supplying power to the other part of the circuit.

Q: Is the current of a voltage source always flowing from + to – terminals?

A: NO. The current of a voltage source is not necessarily flowing from the positive terminal to the negative terminal.

Frequently Asked Questions

Q: What is the voltage cross over a current source?

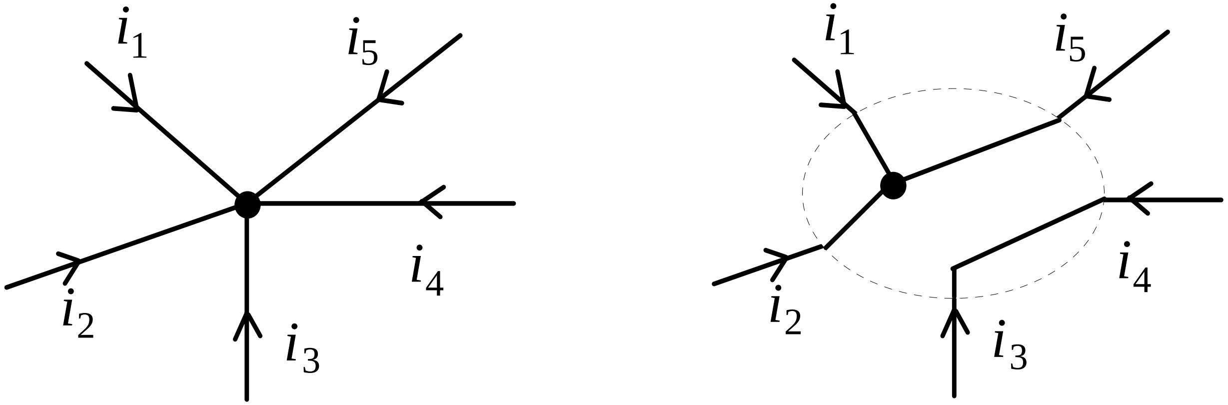
A: It depends on the circuit connected to it.

Q: Is the reference power (or voltage, or current) in the definition of the relative power (or voltage, or current) unique?

A: No. The reference power (voltage or current) can be any value. Note that whenever you deal with the relative power (voltage or current), you should keep in your mind that there are *a reference power* (voltage or current) and *an actual power* (voltage or current) associated with it.

2.6 Kirchhoff's Current Law (KCL)

As demonstrated by the following examples, this states that the algebraic sum of the currents entering/leaving a node/closed surface is 0 or equivalently to say that the total currents flowing into a node is equal to the total currents flowing out from the node.

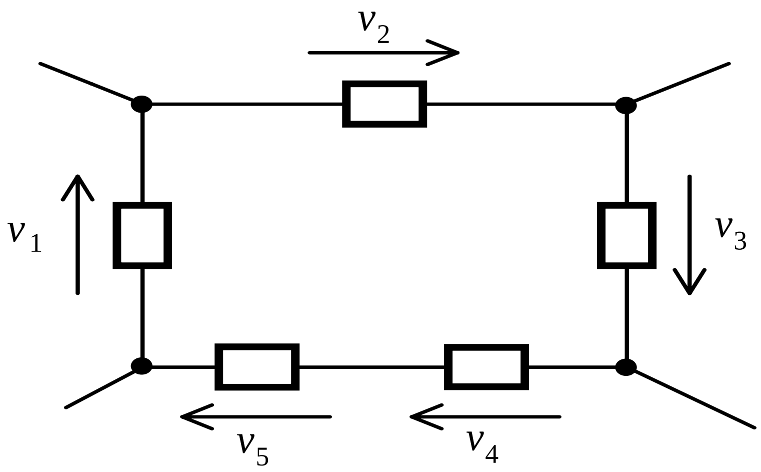


$$i_1 + i_2 + i_3 + i_4 + i_5 = 0$$ for both cases.

Since current is equal to the rate of flow of charges, KCL actually corresponds to the conservation of charges.

2.7 Kirchhoff's Voltage Law (KVL)

As illustrated below, this states that the algebraic sum of the voltage drops around any close loop in a circuit is 0.



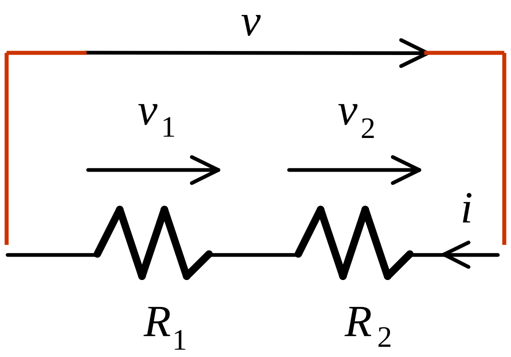
$$v_1+v_2+v_3+v_4+v_5=0$$

(note: all voltages are in the same direction)

Since a charge q will have its electric potential changed by qv_1, qv_2, qv_3, qv_4 , qv_5 as it passes through each of the components, the total energy change in one full loop is $q\,(v_1+v_2+v_3+v_4+v_5)$. Thus, from the conservation of energy: $v_1+v_2+v_3+v_4+v_5=0.$

2.8 Series Circuit

Consider 2 resistors connected in series:



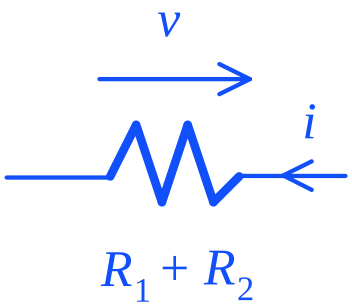
$v_1 = i\,R_1 \qquad v_2 = i\,R_2$

By KVL: $-v + v_1 + v_2 = 0$

$$v = v_1 + v_2$$

the voltage-current relationship is $v = i\left(R_1 + R_2\right)$

Now consider



the voltage-current relationship is $v = i\left(R_1 + R_2\right)$

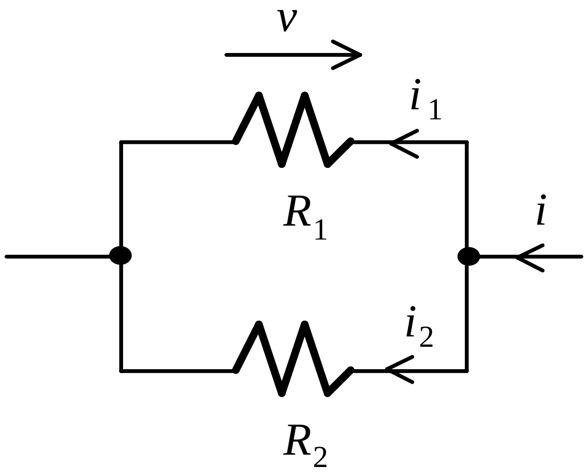
Since the voltage/current relationships are the same for both circuits, they are equivalent from an electrical point of view. In general, for n resistors R_1, ..., R_n connected in series, the equivalent resistance R is

$$R = R_1 + \cdots + R_n$$

Clearly, the resistance's of resistors connected in series add (Prove it).

2.9 Parallel Circuit

Consider 2 resistors connected in parallel:



$$i_1 = \frac{v}{R_1}$$

$$i_2 = \frac{v}{R_2}$$

$$i = i_1 + i_2 = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Clearly, the parallel circuit is equivalent to a resistor *R* with voltage/current relationship

$$i = \frac{v}{R} \qquad \text{with} \qquad \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

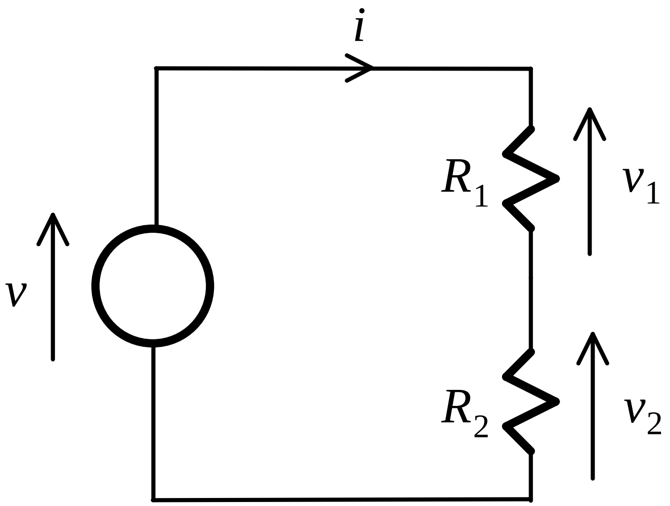
In general, for n resistors R_1, …, R_n, connected in parallel, the equivalent resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \cdots + \frac{1}{R_n}$$

Note that $1/R$ is often called the ***conductance*** of the resistor R. Thus, the conductances of resistors connected in parallel add.

2.10 Voltage Division

Consider 2 resistors connected in series:



$$i = \frac{v}{R_1 + R_2}$$

$$v_1 = iR_1 = \left(\frac{R_1}{R_1 + R_2}\right)v$$

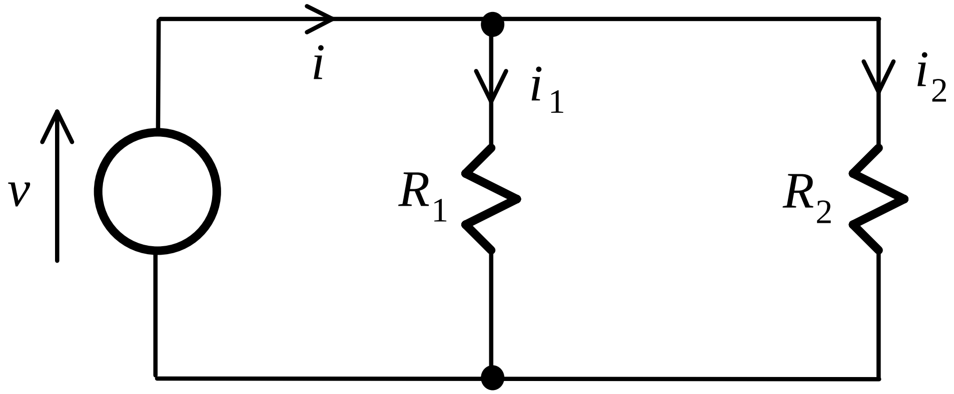
$$v_2 = iR_2 = \left(\frac{R_2}{R_1 + R_2}\right)v$$

The total resistance of the circuit is $R_1 + R_2$. Thus,

$$\boxed{\frac{v_1}{v_2} = \frac{R_1}{R_2}}$$

2.11 Current Division

Consider 2 resistors connected in parallel:



$$i_1 = \frac{v}{R_1} = \left(\frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} \right) i = \left(\frac{R_2}{R_1+R_2} \right) i$$

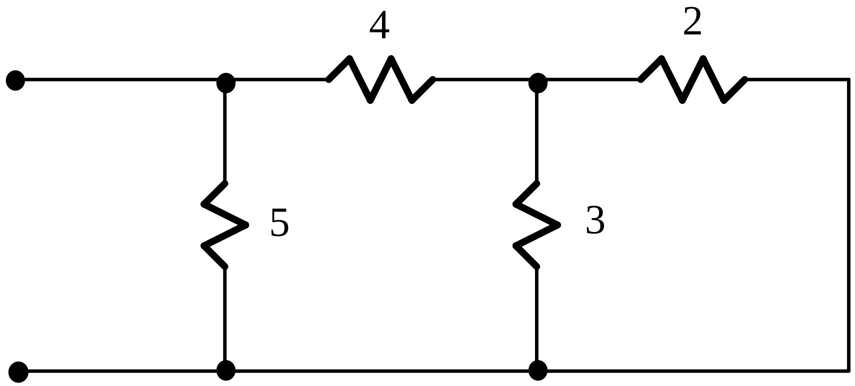
$$i_2 = \frac{v}{R_2} = \left(\frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \right) i = \left(\frac{R_1}{R_1+R_2} \right) i$$

The total conductance of the circuit is $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ Thus,

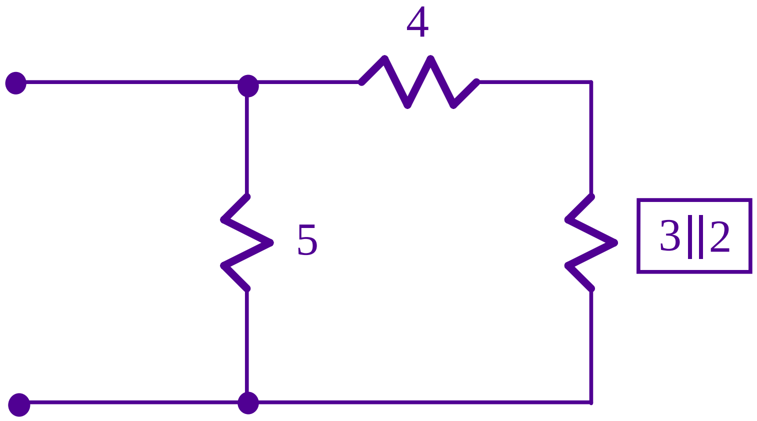
$$\boxed{\frac{i_1}{i_2} = \frac{R_2}{R_1}}$$

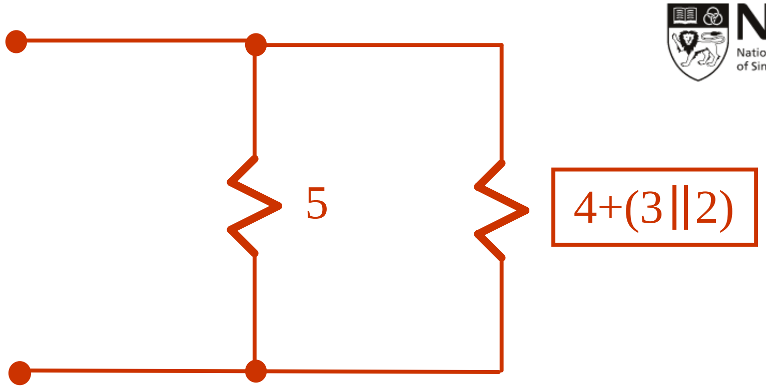
2.12 Ladder Circuit

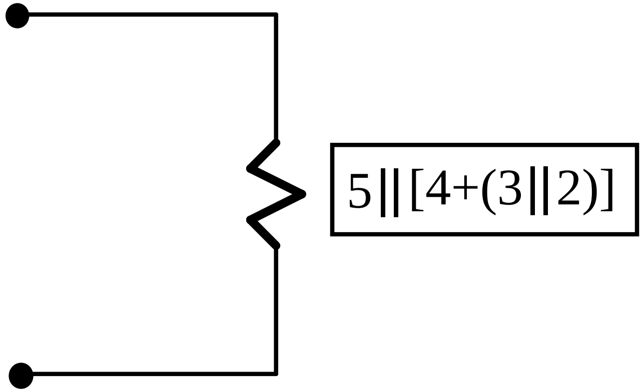
Consider the following ladder circuit:



The equivalent resistance can be determined as follows:





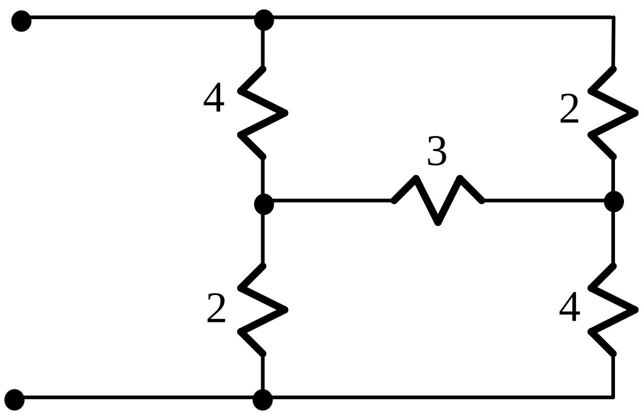


The network is equivalent to a resistor with resistance

$$R = 5 \,||\, [4+(3\,||\,2)] = \frac{1}{\frac{1}{5}+\frac{1}{4+(3\,||\,2)}} = \frac{1}{\frac{1}{5}+\frac{1}{4+\frac{1}{\frac{1}{3}+\frac{1}{2}}}}$$

2.13 Branch Current Analysis

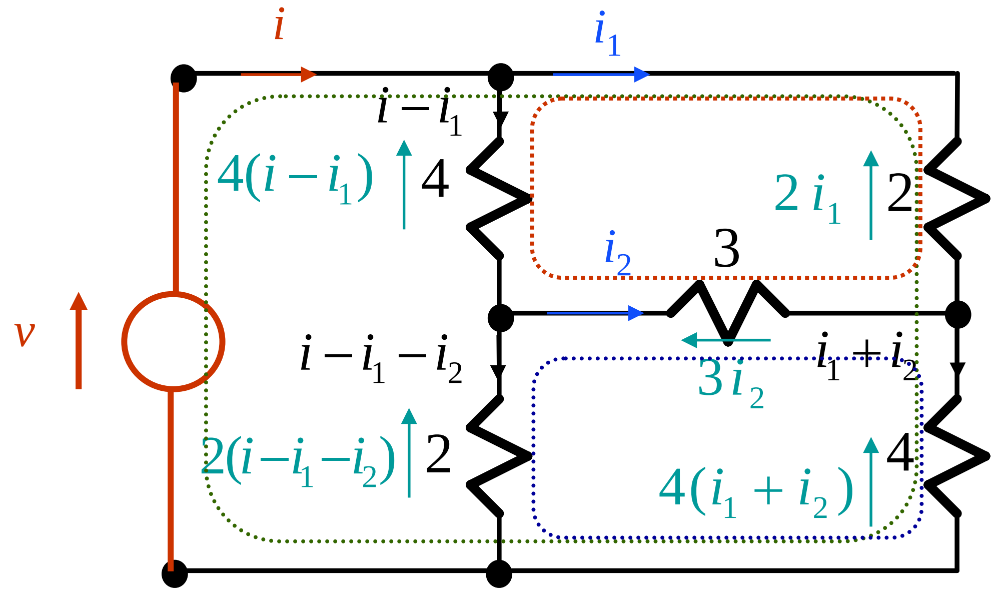
Consider the problem of determining the equivalent resistance of the following ***bridge*** circuit:



Since the components are not connected in straightforward series or parallel manner, it is not possible to use the series or parallel connection rules to simplify the circuit. However, the voltage-current relationship can be determined and this will enables the equivalent resistance to be calculated.

One method to determine the voltage-current relationship is to use the ***branch current method***.

Branch Current Analysis: Example One



1. Assign branch currents (with any directions you prefer so that currents in other branches can be found)

2. Find all other branch currents (with any directions you prefer) (Use KCL to find them)

3. Write down branch voltages

4. Identify independent loops

$$\frac{v}{i} = \text{Equivalent Resistance}$$

KVL: $v = 2i_1 + 4(i_1 + i_2) = 6i_1 + 4i_2$

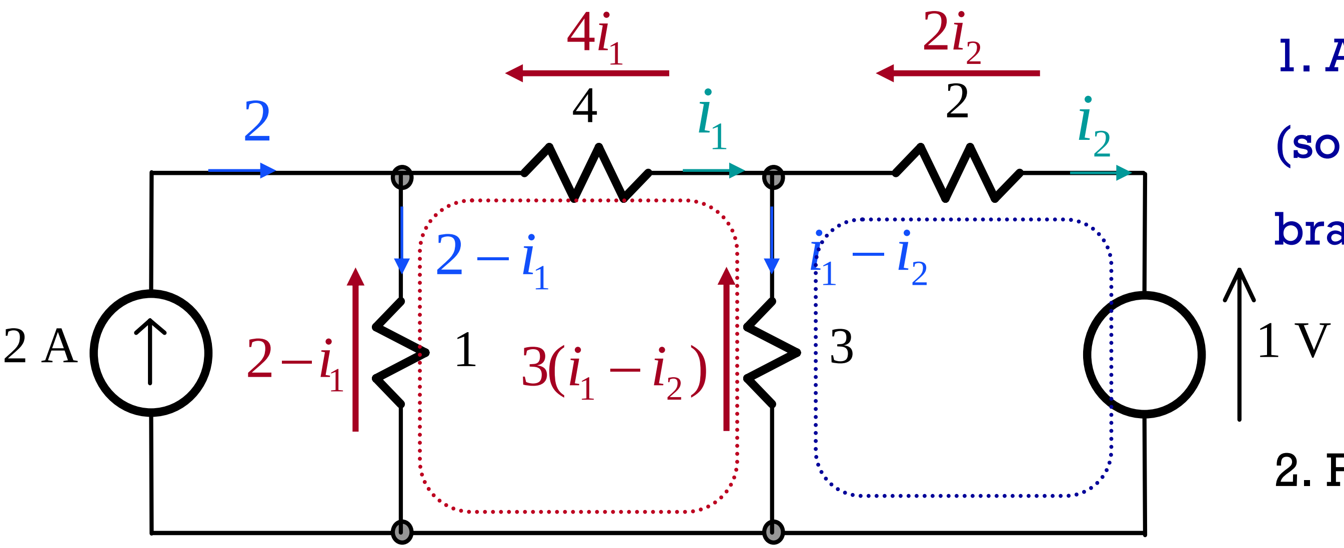
KVL: $4(i - i_1) + 3i_2 = 2i_1$

KVL: $4(i_1 + i_2) + 3i_2 = 2(i - i_1 - i_2)$

Eliminate i_1 and i_2,

$$i_1 = \frac{7i}{12} \qquad i_2 = -\frac{i}{6} \qquad \boxed{v = \frac{17}{6}i}$$

Branch Current Analysis: Example Two



1. Assign branch currents (so that currents in other branches can be found).

2. Find all other branch currents (KCL)

3. Write down voltages across components

KVL: $2-i_1=4i_1+3(i_1-i_2)$

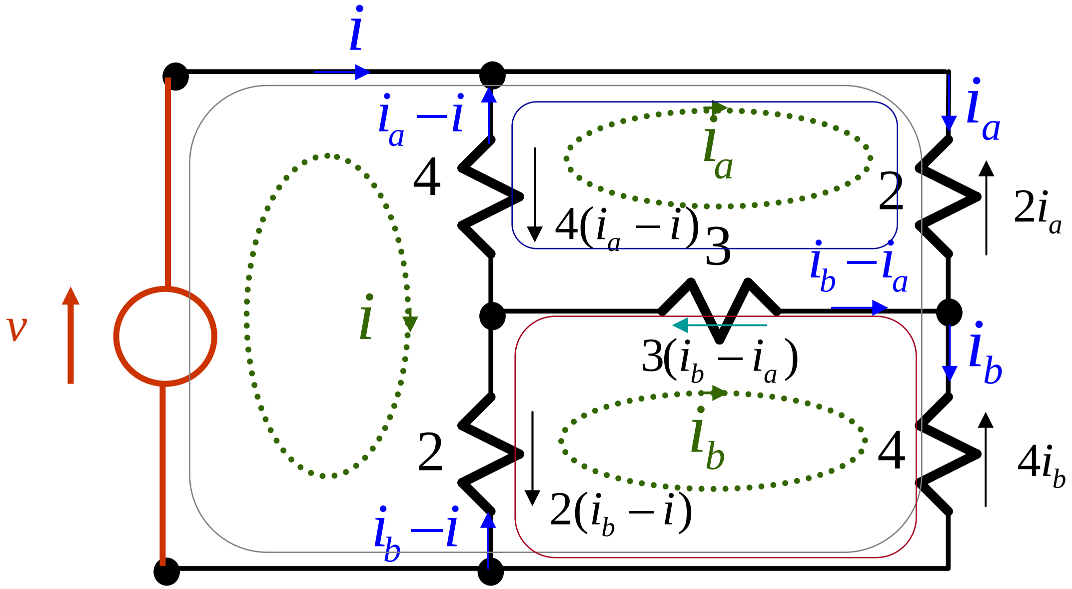
4. Identify independent loops (ex. 2A branch)

KVL: $1+2i_2=3(i_1-i_2)$

This implies:

$$\begin{aligned}3i_1-5i_2&=1\\8i_1-3i_2&=2\end{aligned}\quad\Rightarrow\quad\begin{bmatrix}3&-5\\8&-3\end{bmatrix}\begin{bmatrix}i_1\\i_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}\quad\Rightarrow\quad\begin{bmatrix}i_1\\i_2\end{bmatrix}=\begin{bmatrix}3&-5\\8&-3\end{bmatrix}^{-1}\begin{bmatrix}1\\2\end{bmatrix}=\frac{1}{31}\begin{bmatrix}7\\-2\end{bmatrix}$$

2.14 Mesh (Loop Current) Analysis



1. Assign fictitious loop currents
2. Find branch currents (KCL)
3. Write down branch voltages
4. Identify independent loops
5. Simplify the equations obtained, we get

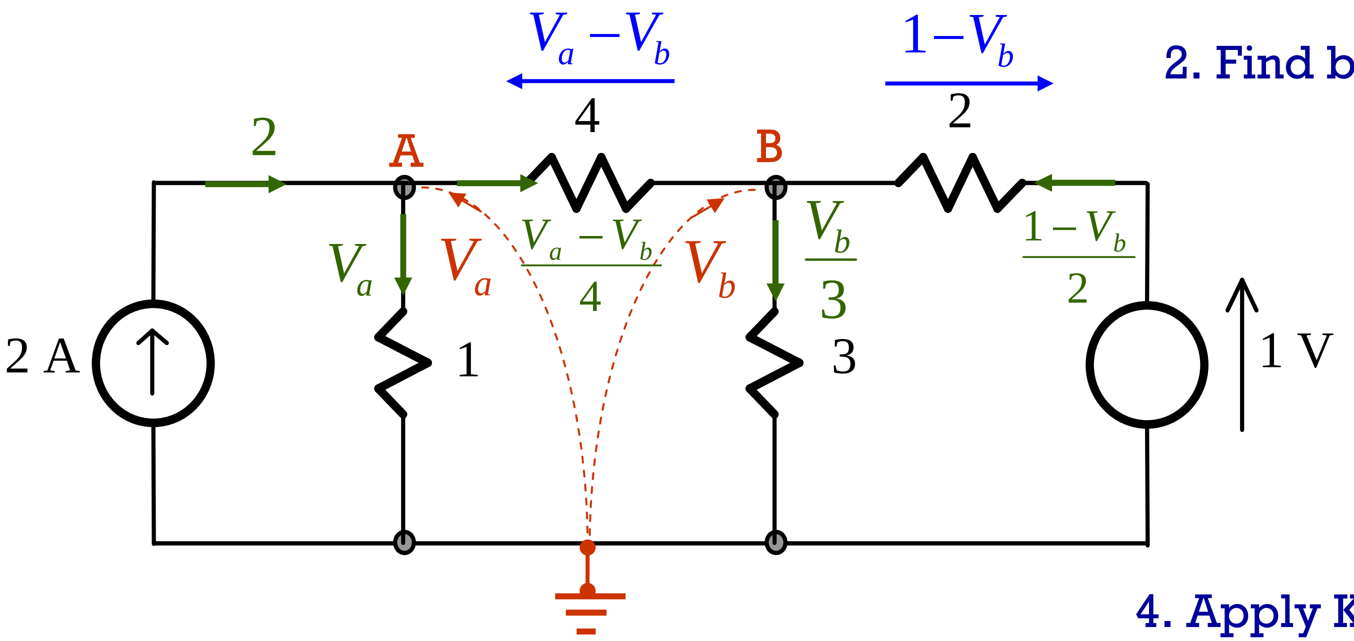
KVL: $v = 2i_a + 4i_b$

KVL: $4(i_a - i) - 3(i_b - i_a) + 2i_a = 0$

KVL: $2(i_b - i) + 4i_b + 3(i_b - i_a) = 0$

$$v = \frac{17i}{6} \quad \Rightarrow \quad R_{equivalence} = \frac{17}{6}$$

2.15 Nodal Analysis



2. Find branch voltages (KVL)

3. Determine

branch currents

4. Apply KCL to Nodes A & B

1. Assign nodal voltage w.r.t. the reference node

$$\begin{bmatrix} 5 & -1 \\ 3 & -13 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix} \quad \Leftarrow \quad \begin{aligned} 5V_a - V_b &= 8 \\ 3V_a - 13V_b &= -6 \end{aligned} \quad \Leftarrow$$

Node A:

$$2 = V_a + \frac{V_a - V_b}{4}$$

Node B:

$$\frac{V_a - V_b}{4} + \frac{1 - V_b}{2} = \frac{V_b}{3}$$

$$\boxed{\Rightarrow \quad V_a = \frac{55}{31} \quad V_b = \frac{27}{31}}$$

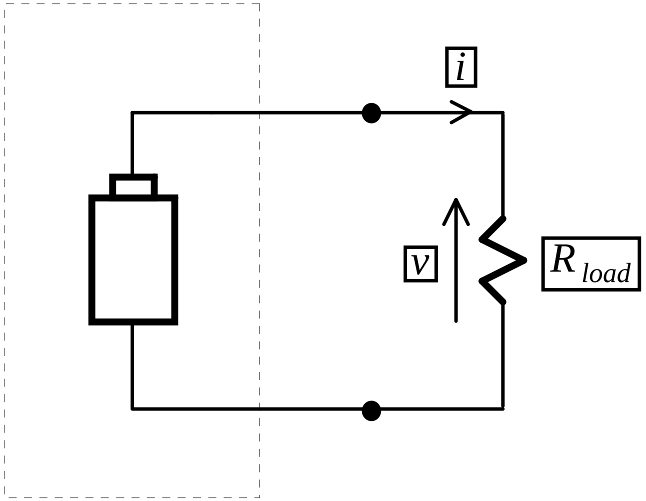
2.16 Practical Voltage Source

An ideal voltage source is one whose terminal voltage does not change with the current drawn. However, the terminal voltages of practical sources usually decrease slightly as the currents drawn are increased.

A commonly used model for a practical voltage source is:

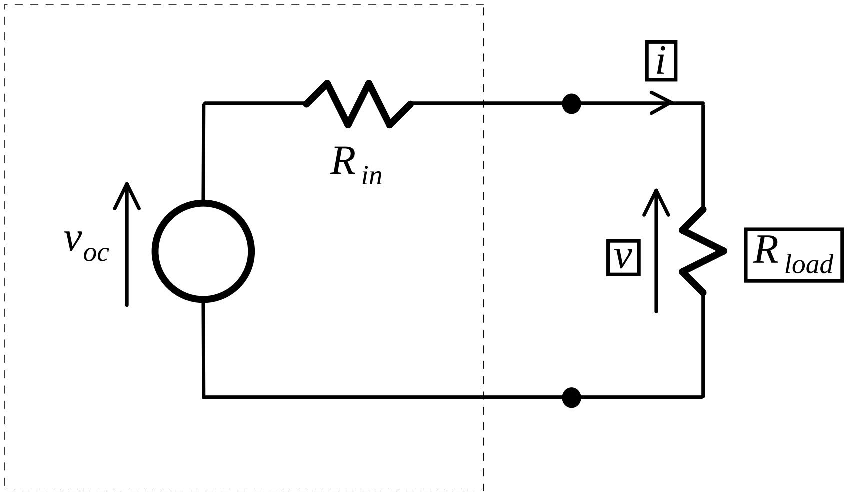
To represent it as a series of an ideal voltage source & an internal resistance.

$$\boxed{v_{oc} = v + iR_{in}} \quad \Leftarrow$$



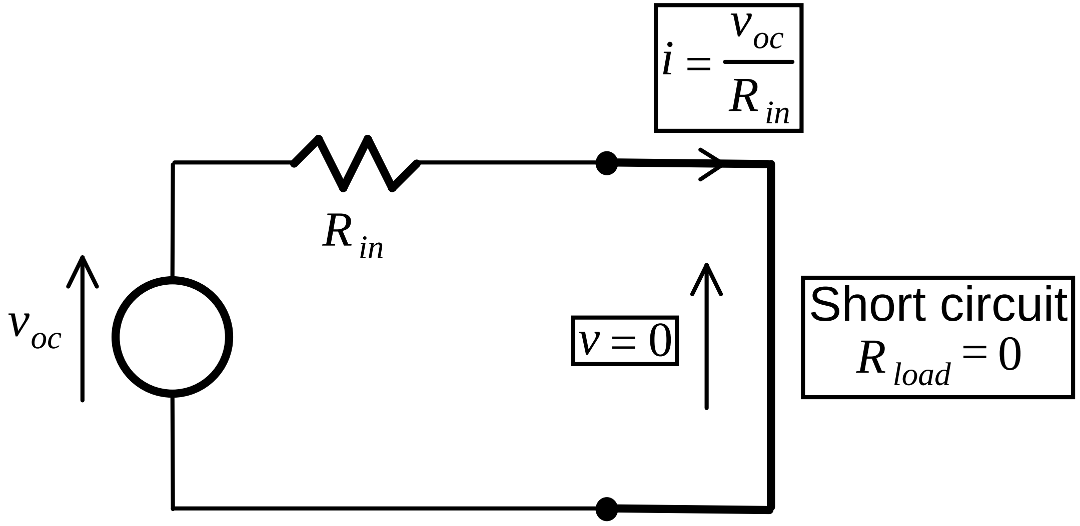
Practical voltage source

|||

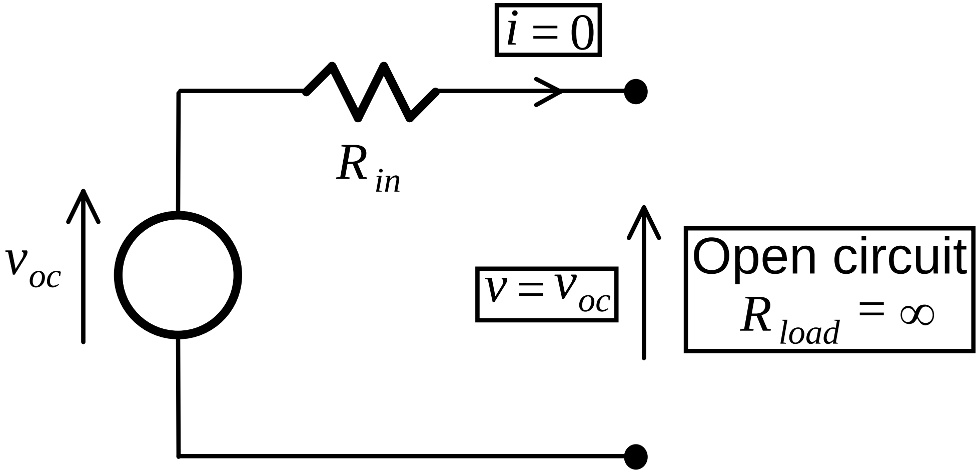


Model for voltage source

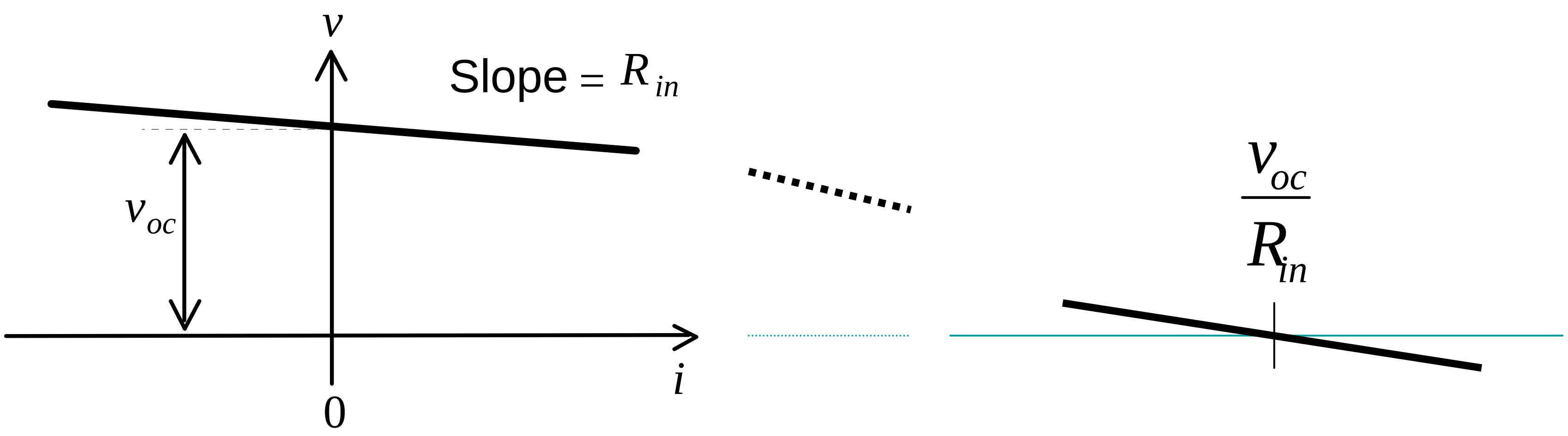
When R_{load} or when the source is ***short circuited*** so that $v = 0$:



When $R_{load} = \infty$ or equivalently when the source is ***open circuited*** so that $i = 0$



Graphically:

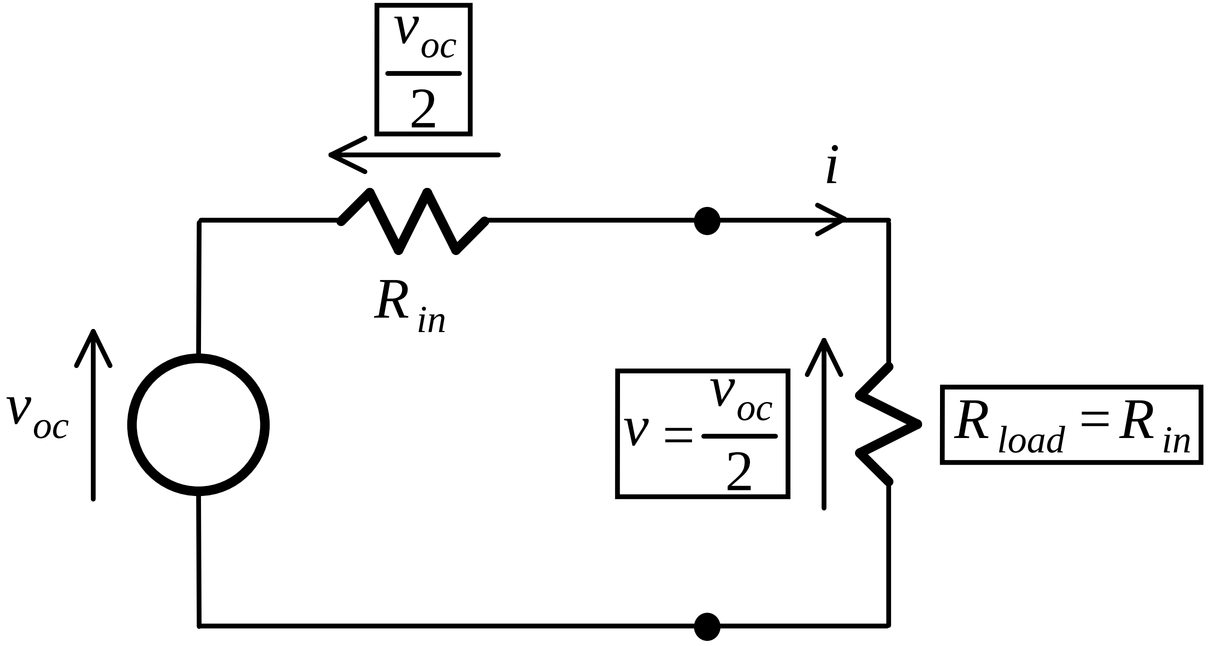


Good practical voltage source should therefore have small ***internal resistance***, so that its output voltage will not deviate very much from the ***open circuit voltage***, under any operating condition.

The internal resistance of an ideal voltage source is therefore zero so that v does not change with i.

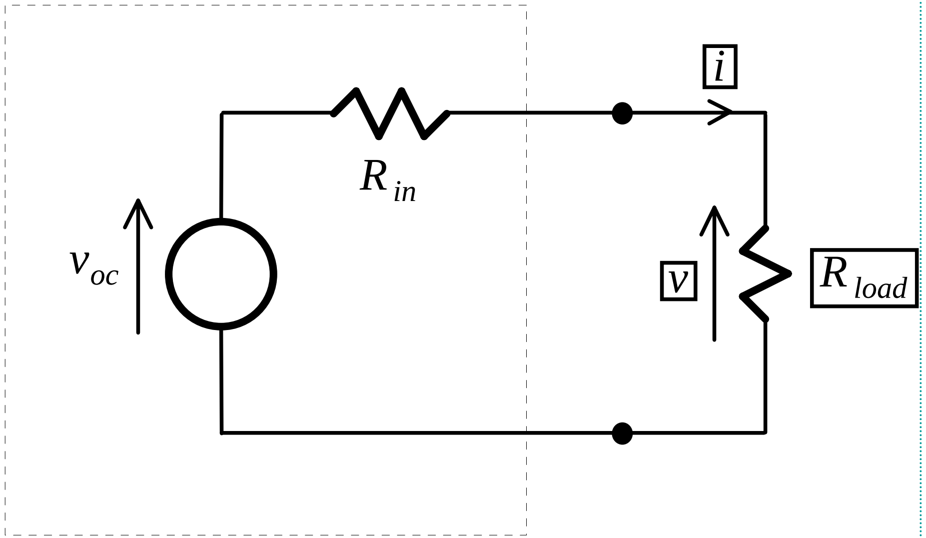
To determine the two parameters v_{oc} and R_{in} that characterize, say, a battery, we can measure the output voltage when the battery is open-circuited (nothing connected except the voltmeter). This will give v_{oc}.

Next, we can connect a load resistor and vary the load resistor such that the voltage across it is $v_{oc} / 2$. The load resistor is then equal to R_{in} :



2.17 Maximum Power Transfer

Consider the following circuit:



Model for voltage source

The current in the load resistor is

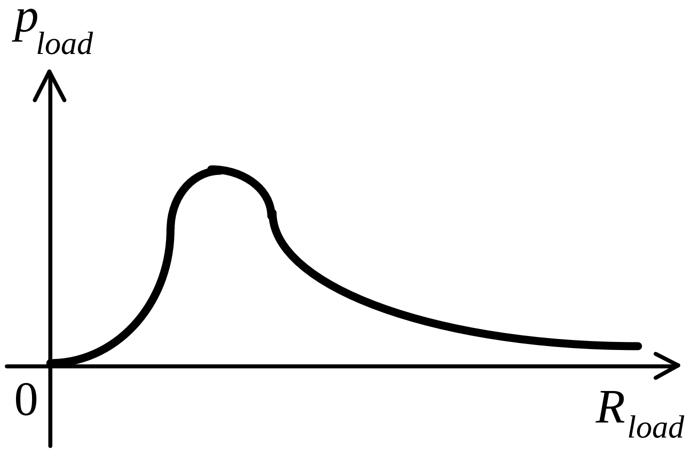
$$i=\frac{v_{oc}}{R_{in}+R_{load}}$$

The power absorbed by the load resistor is

$$p_{load}=i^2 R_{load}=\frac{v_{oc}^2 R_{load}}{\left(R_{in}+R_{load}\right)^2}$$

This is always positive. However, if

$R_{load}=0$ or $R_{load}=\infty$, $P_{load}=0$.

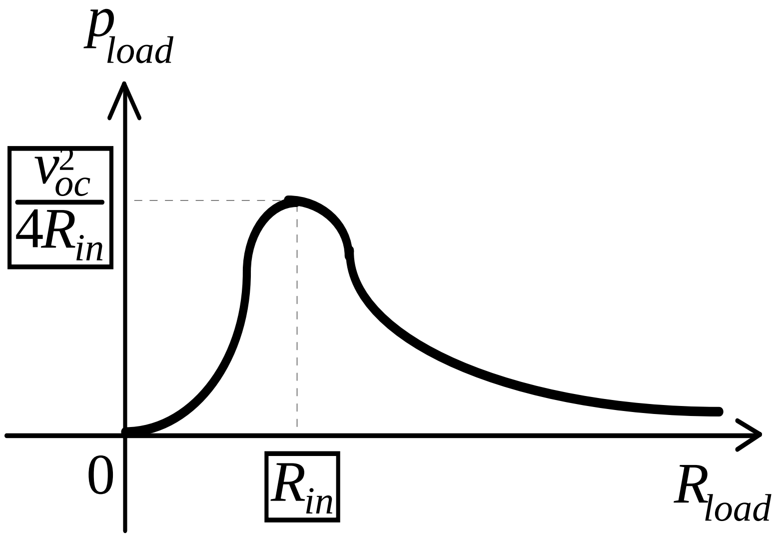


Differentiating:

$$\frac{dp_{load}}{dR_{load}} = v_{oc}^2\left[\frac{1}{(R_{in}+R_{load})^2} - \frac{2R_{load}}{(R_{in}+R_{load})^3}\right] = v_{oc}^2\left[\frac{R_{in}-R_{load}}{(R_{in}+R_{load})^3}\right]$$

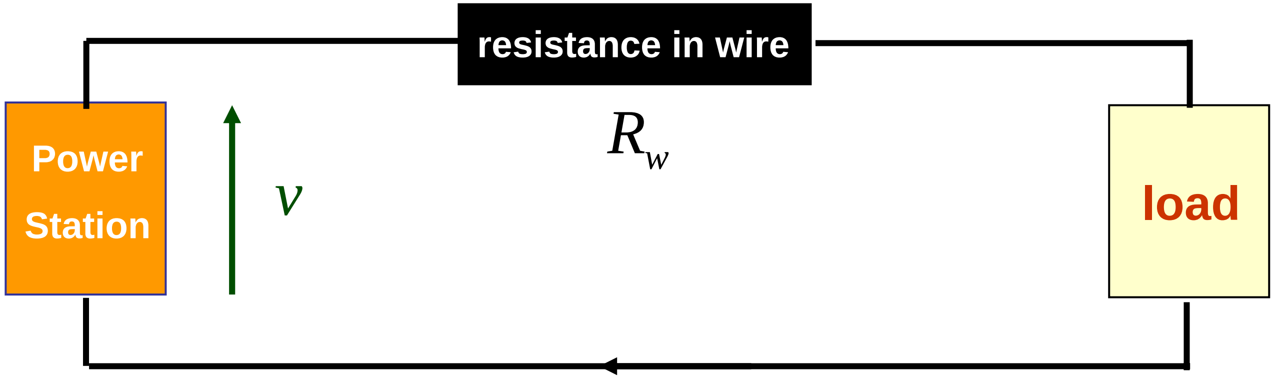
The load resistor will be absorbing the maximum power or the source will be transferring the maximum power if the load and source internal resistances are matched, i.e., $R_{in} = R_{load}$. The maximum power transferred is given by

$$p_{load} = i^2 R_{load} = \frac{v_{oc}^2 R_{load}}{(R_{in}+R_{load})^2} \quad \Rightarrow \quad p_{\text{max load}} = \frac{v_{oc}^2}{4R_{in}}$$



When the load absorbs the maximum power from the source, the overall power efficiency of 50%, which is too low for a usual electric system.

Why is the electric power transferred from power stations to local stations in high voltages?



$$P = 300\,\text{KW} \qquad i = {300\,\text{KW}}/{v}$$

Power loss in the transmission line:

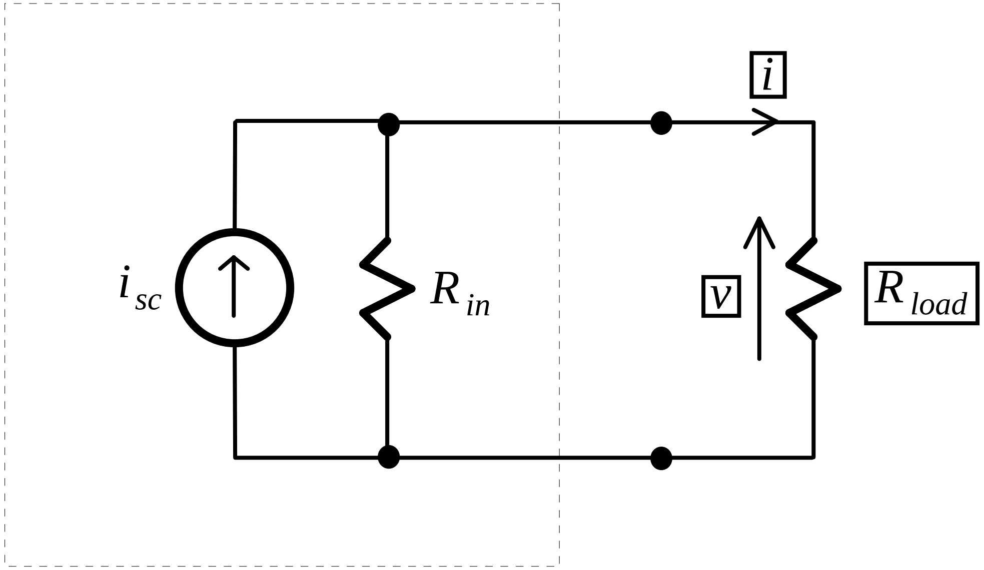
$$P_{loss} = i^2 R_w = \frac{(300\,\text{KW})^2 R_w}{v^2}$$

The higher voltage v is transmitted, the less power is lost in the wire.

2.18 Practical Current Source

An ideal current source is one which delivers a constant current regardless of its terminal voltage. However, the current delivered by a practical current source usually changes slightly depending on the load and the terminal voltage.

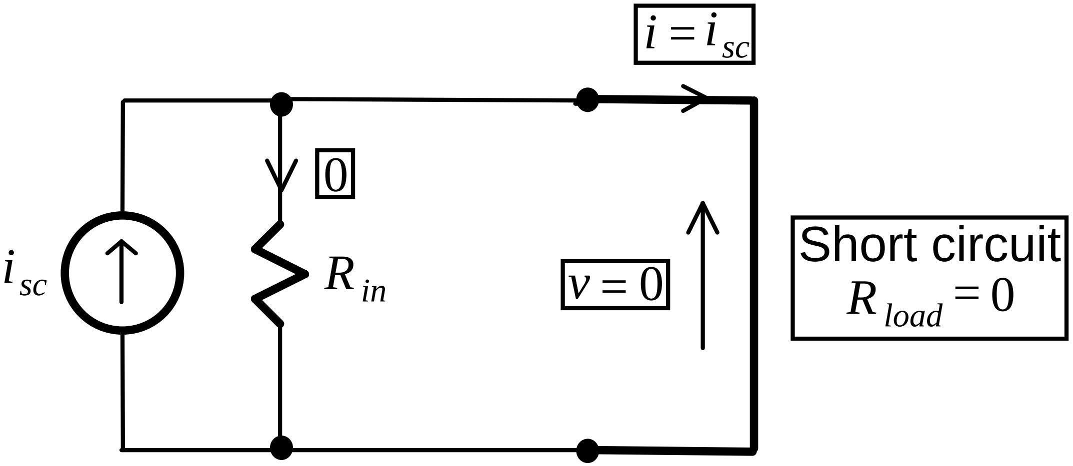
A commonly used model for a current source is:



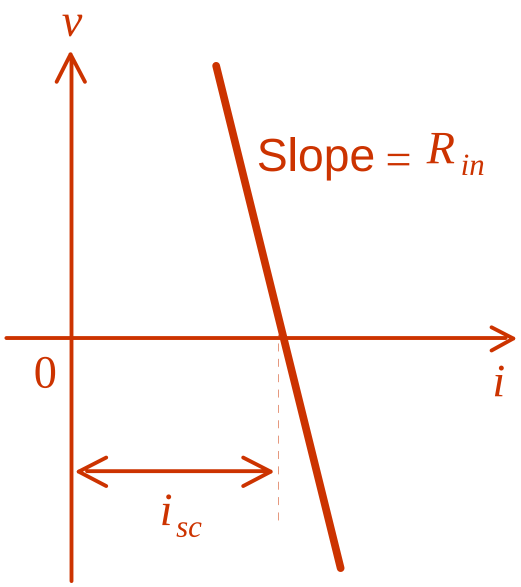
$$\Rightarrow \qquad i_{sc} = \frac{v}{R_{in}} + i$$

Model for current source

When $R_{load} = 0$ or when the source is short-circuited so that $v = 0$:



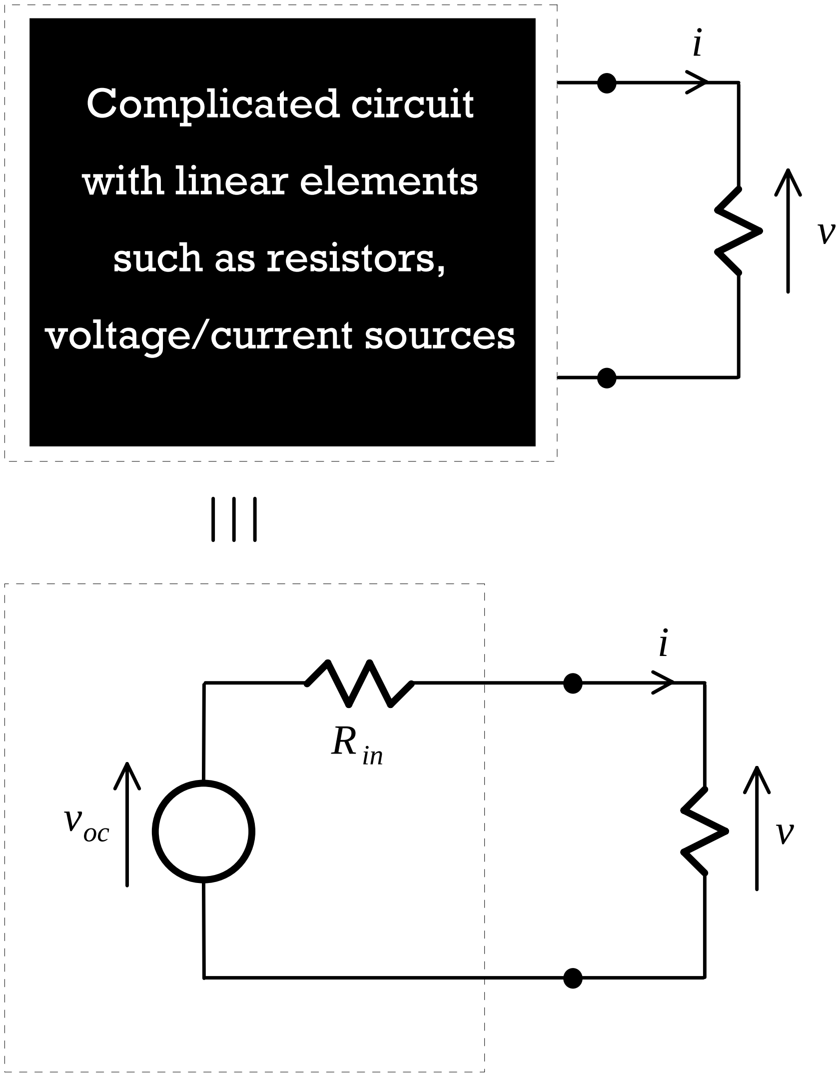
Graphically:



Good practical current source should therefore have large internal resistance so that the current delivered does not deviate very much from the ***short circuit current*** under any operating condition.

The internal resistance of an ideal current source is therefore infinity so that i does not change with v.

2.19 Thevenin's Equivalent Circuit


Complicated circuit with linear elements such as resistors, voltage/current sources

$$v_{oc} = v + i R_{in}$$

$$v_{oc} = v + i R_{in}$$

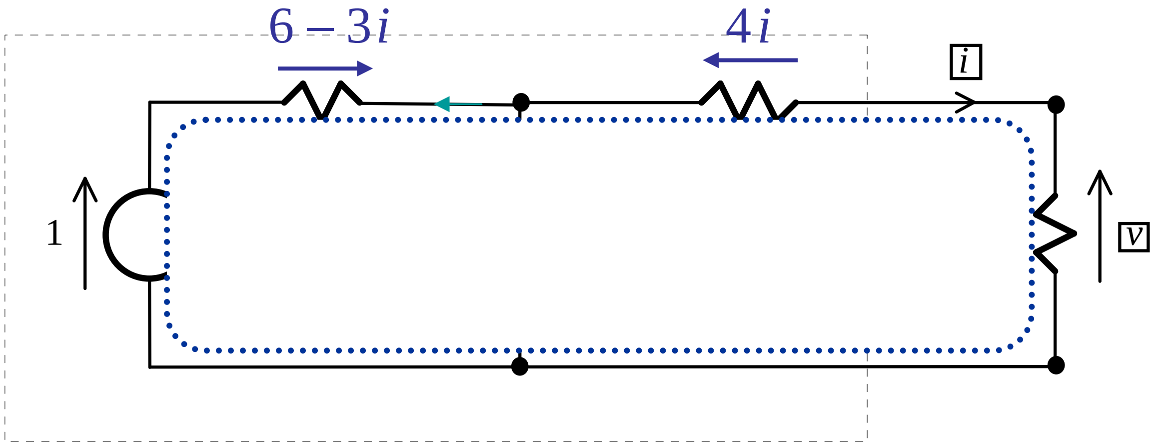
Thevenin's equivalent circuit

Key points:

1. The black box (i.e., the part of the circuit) to be simplified must be linear.

2. The black box must have two terminals connected to the rest of the circuit.

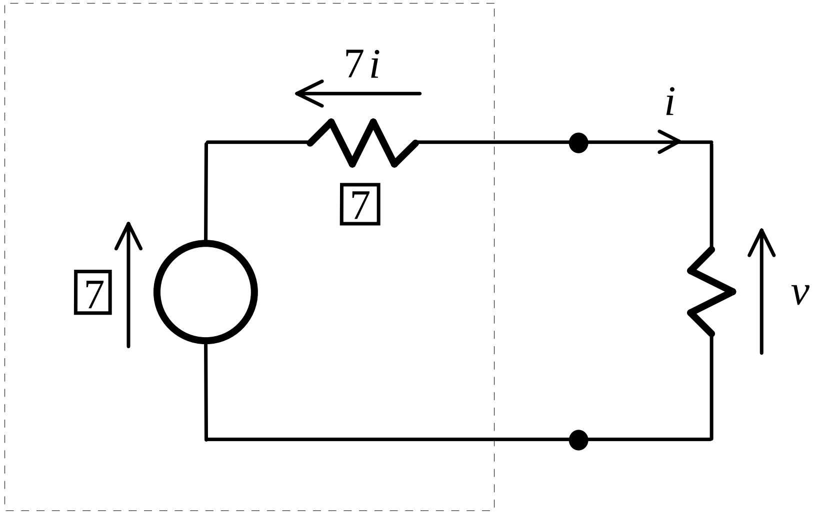
Thevenin’s Equivalent Circuit (An Example)



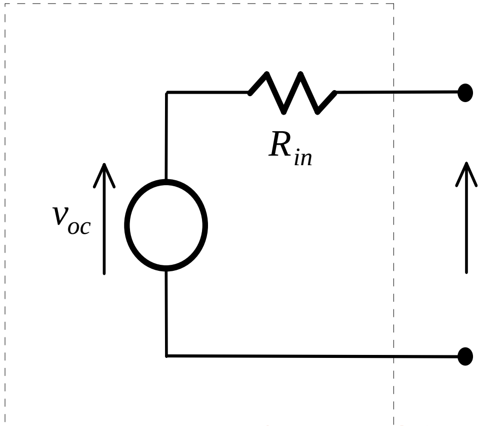
Applying KVL: $1+(6-3i) = 4i + v \quad \Rightarrow \quad 7 = v + 7i$

The circuit is equivalent to:

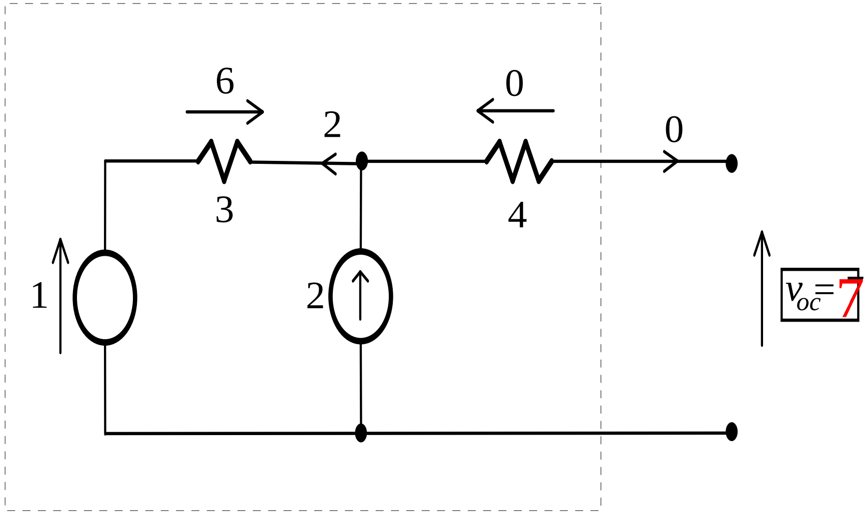
$$\boxed{v_{oc} = 7 \quad R_{in} = 7} \quad \Leftarrow$$



Alternatively, note that from the Thevenin's equivalent circuit:

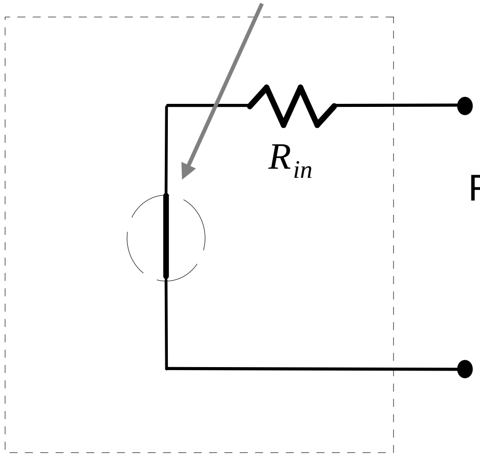


Open circuit voltage= v_{oc}

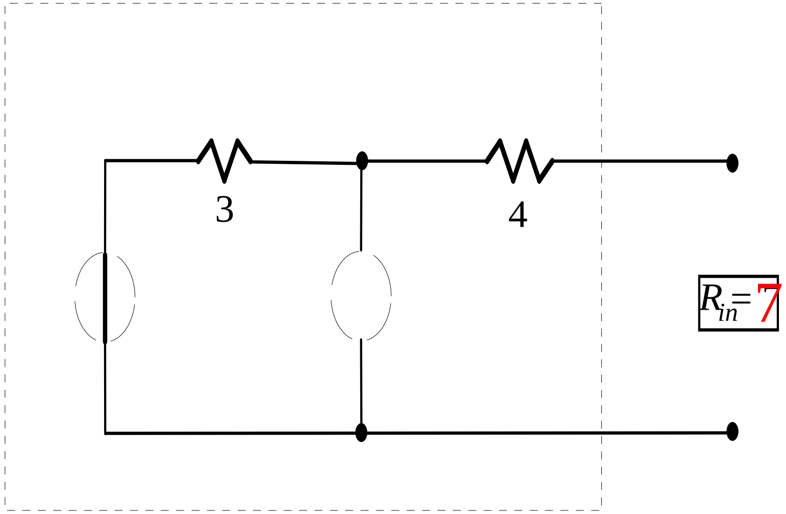


Short Circuit

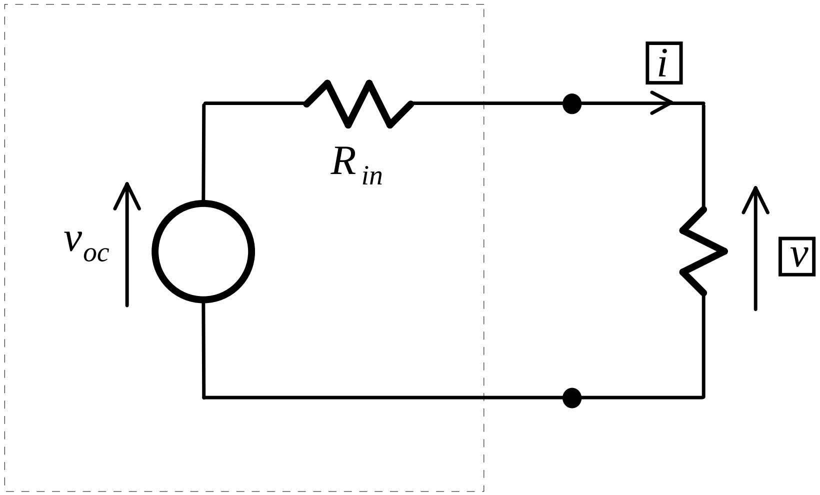
$\Longrightarrow$



Resistance seen with
source replace by= R_{in}
internal resistance

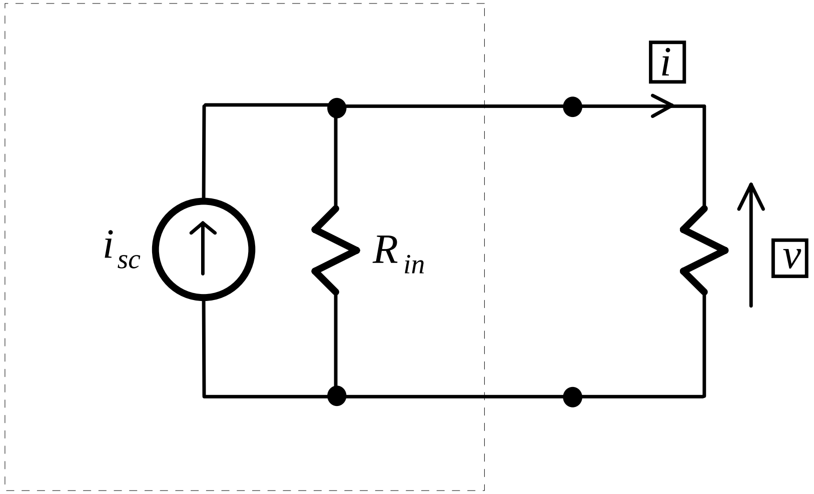


2.20 Norton's Equivalent Circuit



Thevenin's equivalent circuit

||| if $v_{oc} = i_{sc}R_{in}$



Norton's equivalent circuit

$$v_{oc} = v + iR_{in}$$

||| if $v_{oc} = i_{sc}R_{in}$

$$i_{sc} = \frac{v}{R_{in}} + i$$

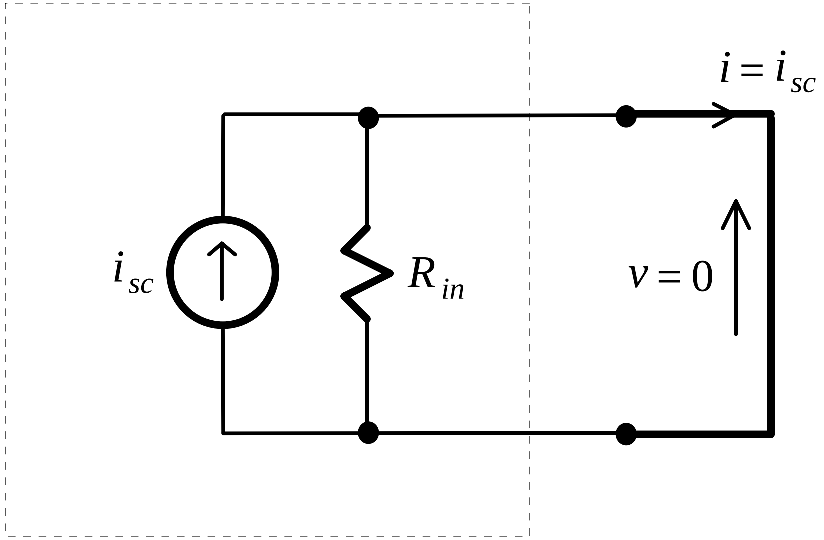
$$i_{sc}R_{in} = v + iR_{in}$$

It is simple to see that if we let

$$v_{oc} = i_{sc}R_{in}$$

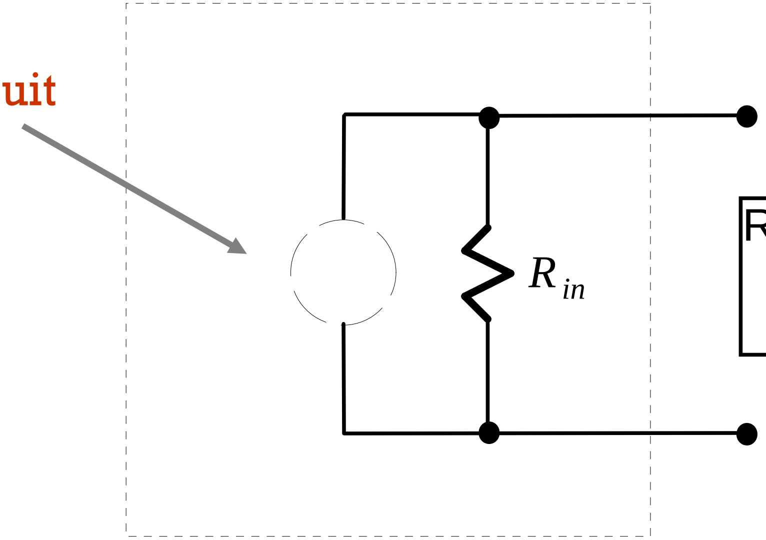
then relationships of voltage/current for both Thevenin’s and Norton’s equivalent circuits are exactly the same.

From the Norton's equivalent circuit, the two parameters i_{sc} and R_{in} can be obtained from:



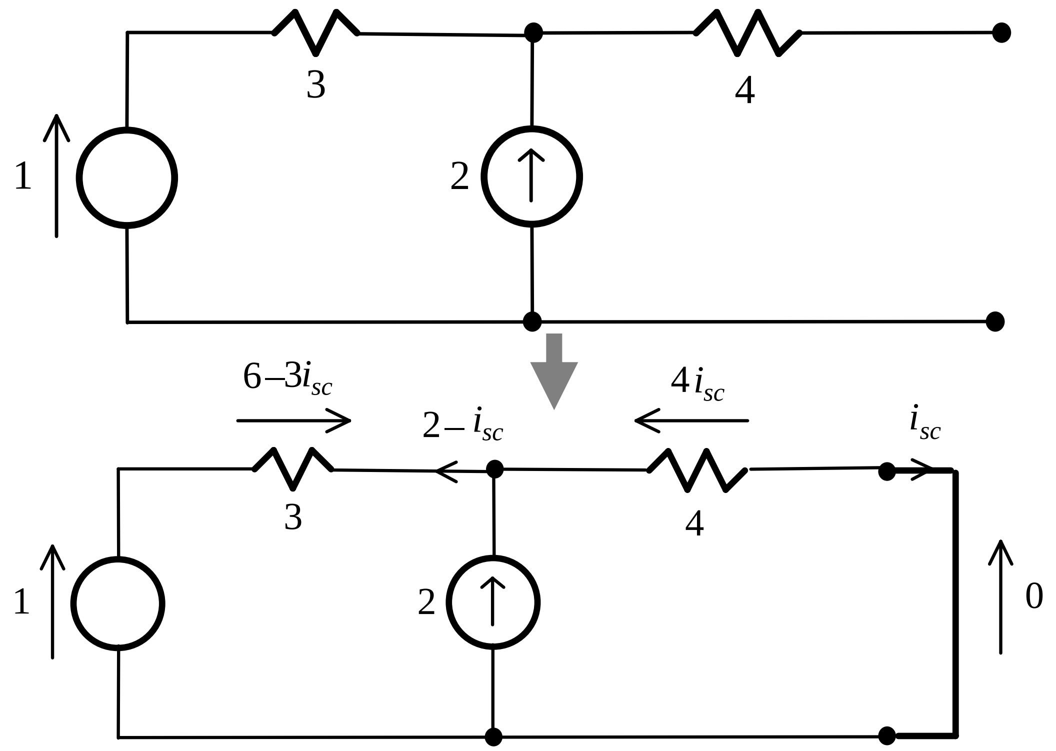
$\boxed{\text{Short circuit current } = i_{sc}}$

Open Circuit

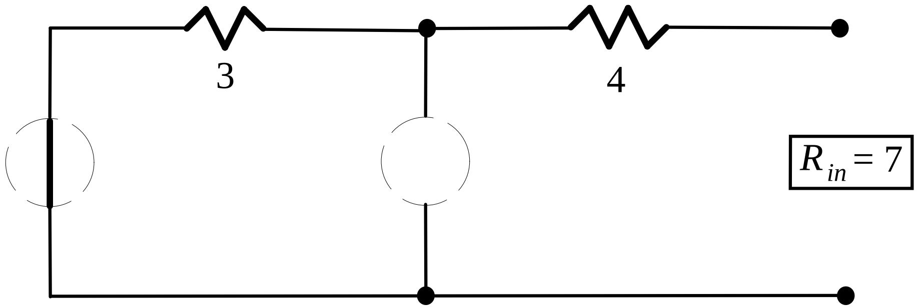


Resistance seen with source replace by internal resistance $= R_{in}$

Example: Reconsider the circuit

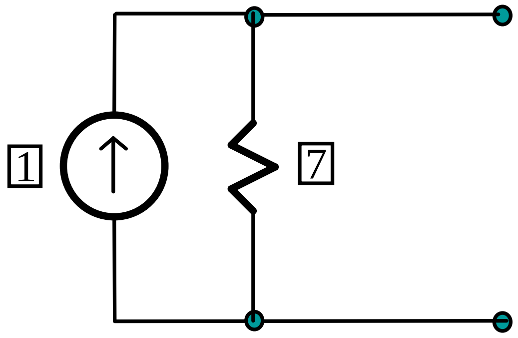


$$\boxed{1+6-3i_{sc}=4i_{sc}\text{ or }i_{sc}=1}$$

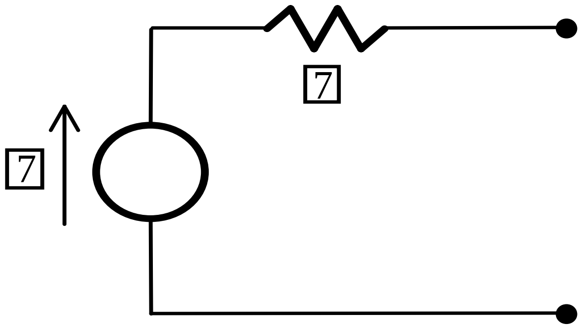


$$\boxed{R_{in}=7}$$

The Norton's equivalent circuit is therefore:



And the Thevenin's equivalent circuit is:



Summary on how to find an equivalent circuit:

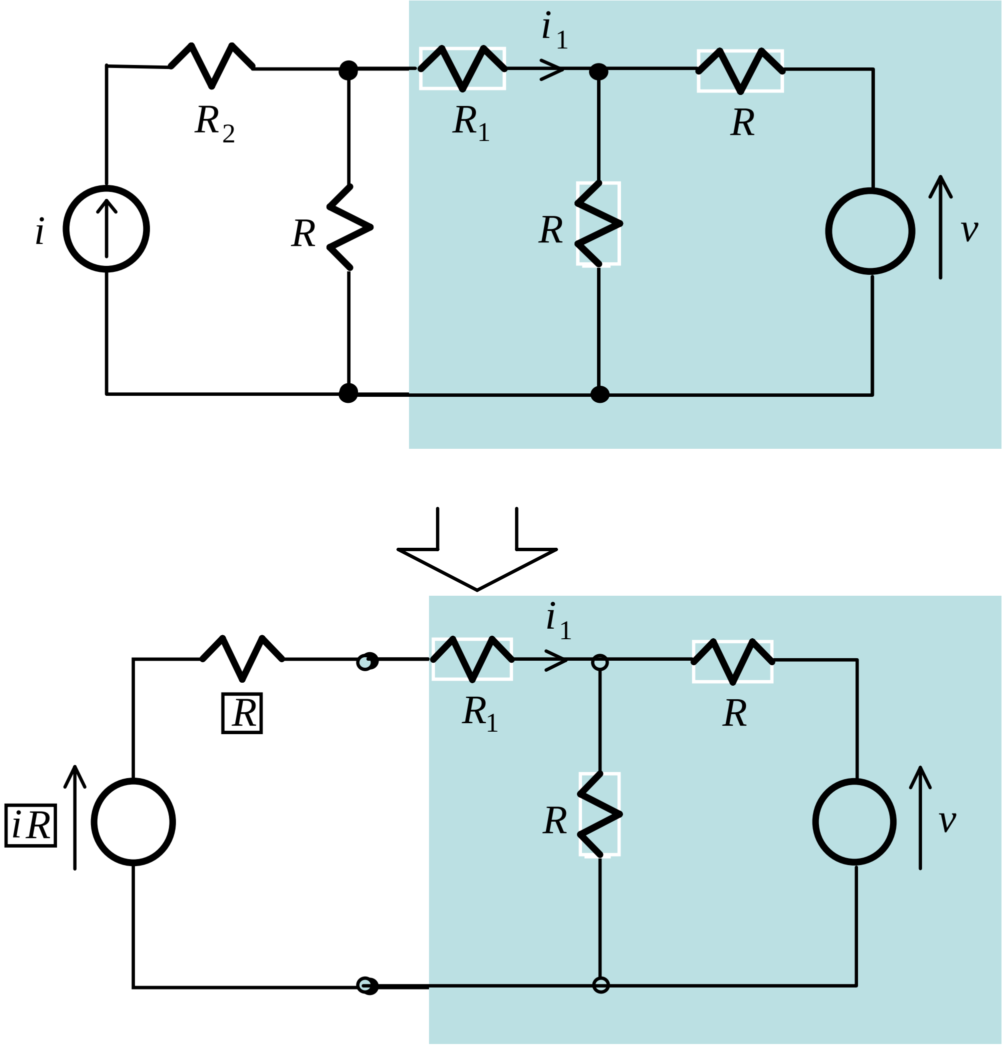
Step 1. Identify the circuit or a portion of a complicated circuit that is to be simplified. Be clear in your mind on which two terminals are to be connected to the other network.

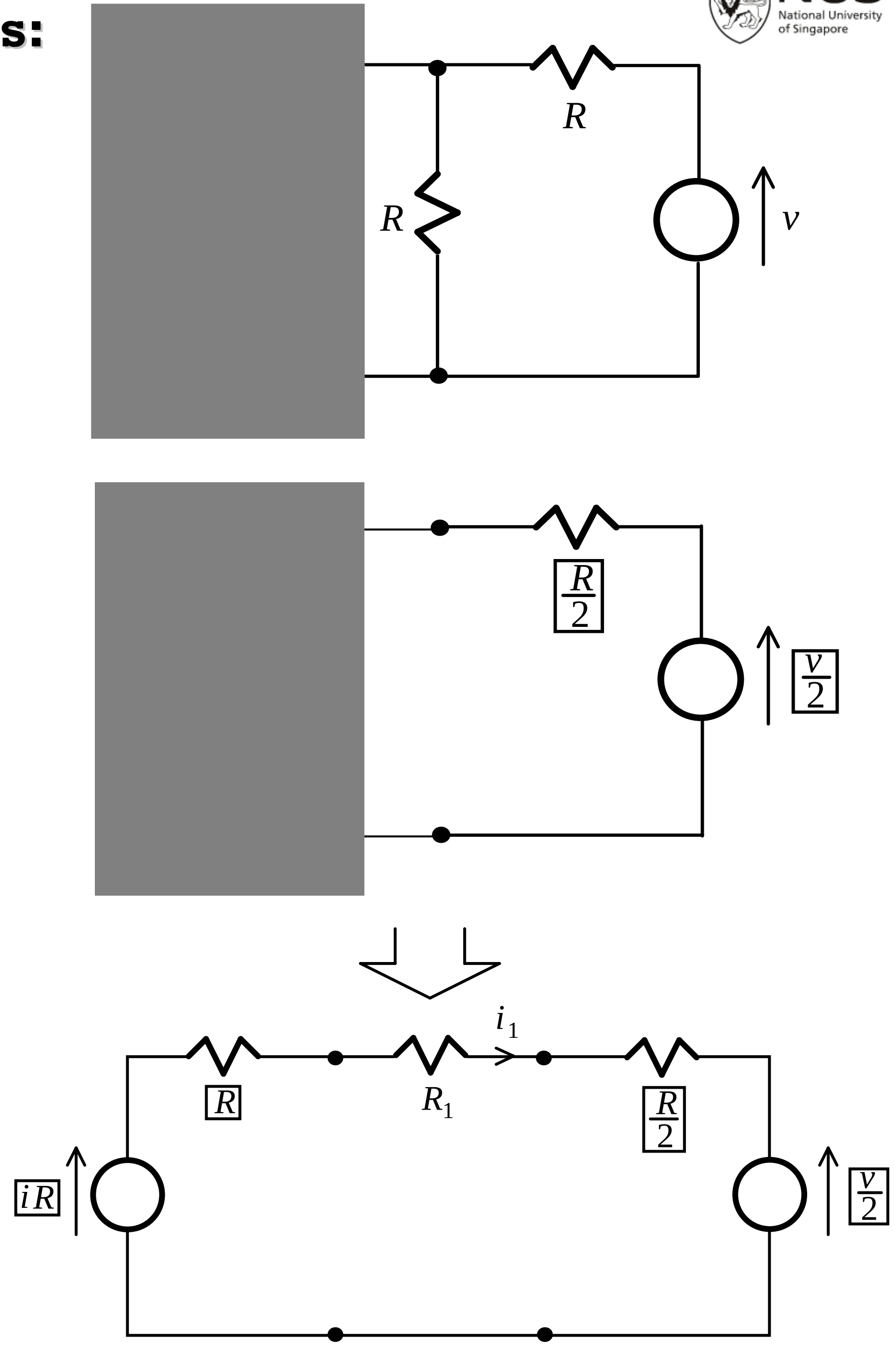
Step 2. Short-circuit all the independent voltage sources and open-circuit all independent current sources in the circuit that you are going to simplify. Then, find the equivalent resistance w.r.t. the two terminals identified in Step 1.

Step 3. Find the open circuit voltage at the output terminals (for Thevenin’s equivalent circuit) or the short circuit current at the output terminals (for Norton’s equivalent circuit).

Step 4. Draw the equivalent circuit (either the Thevenin’s or Norton’s one).

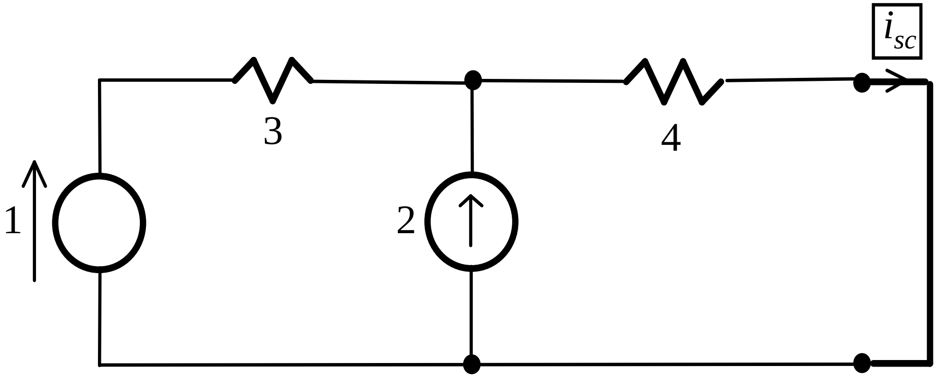
More Example For Equivalent Circuits:



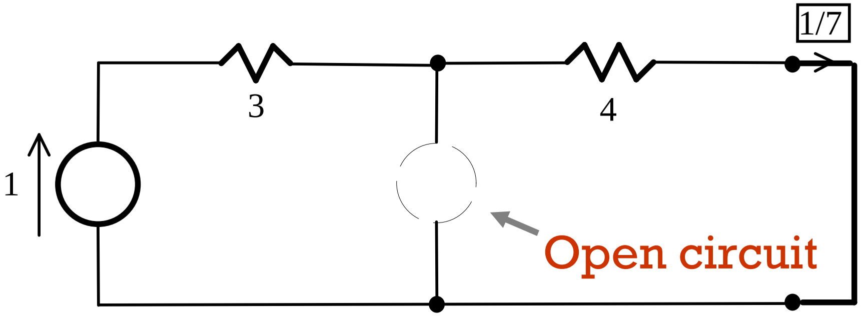


2.21 Superposition

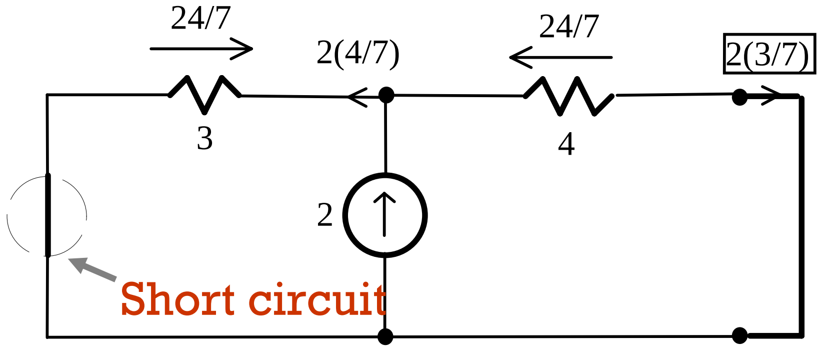
Consider finding i_{sc} in the circuit:



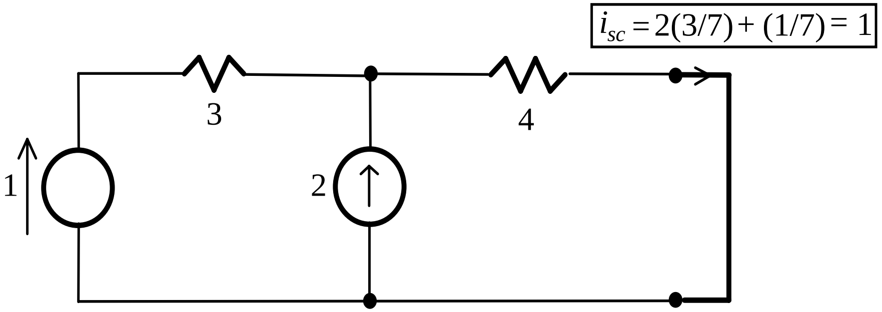
By using the principle of superposition, this can be done by finding the components of i_{sc} due to the 2 ***independent*** sources on their own (with the other sources replaced by their internal resistances):



Open circuit

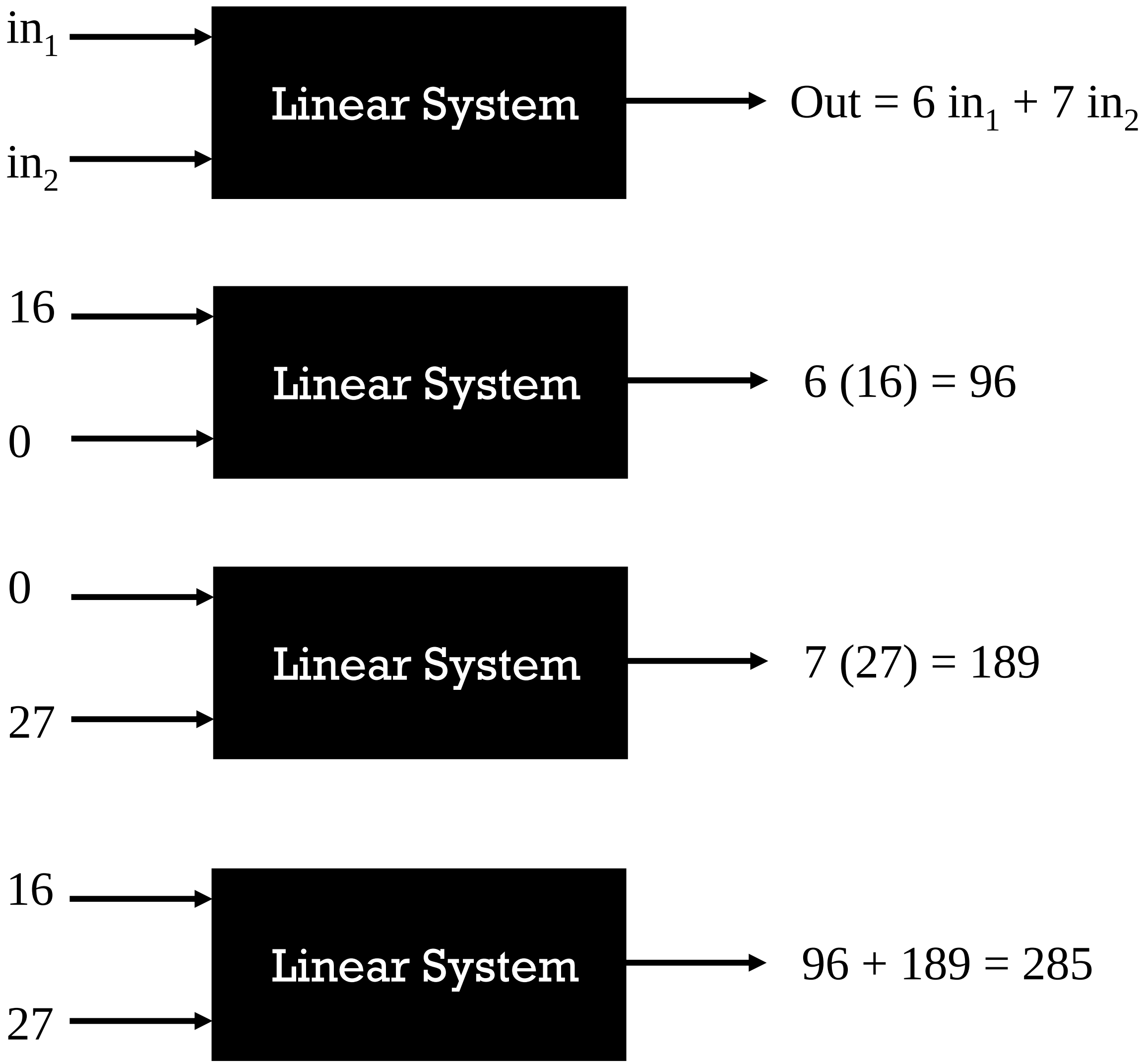


Short circuit



$$i_{sc} = 2(3/7) + (1/7) = 1$$

Linear Systems and Superposition



in_1, in_2 → Linear System → $Out = 6\ in_1 + 7\ in_2$

(the definition of the linear system)

16, 0 → Linear System → $6\ (16) = 96$

0, 27 → Linear System → $7\ (27) = 189$

16, 27 → Linear System → $96 + 189 = 285$

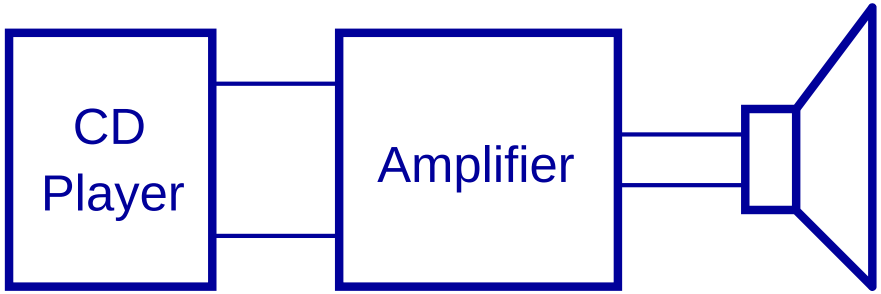
Note that:

Linear system: linear relationship between inputs and outputs

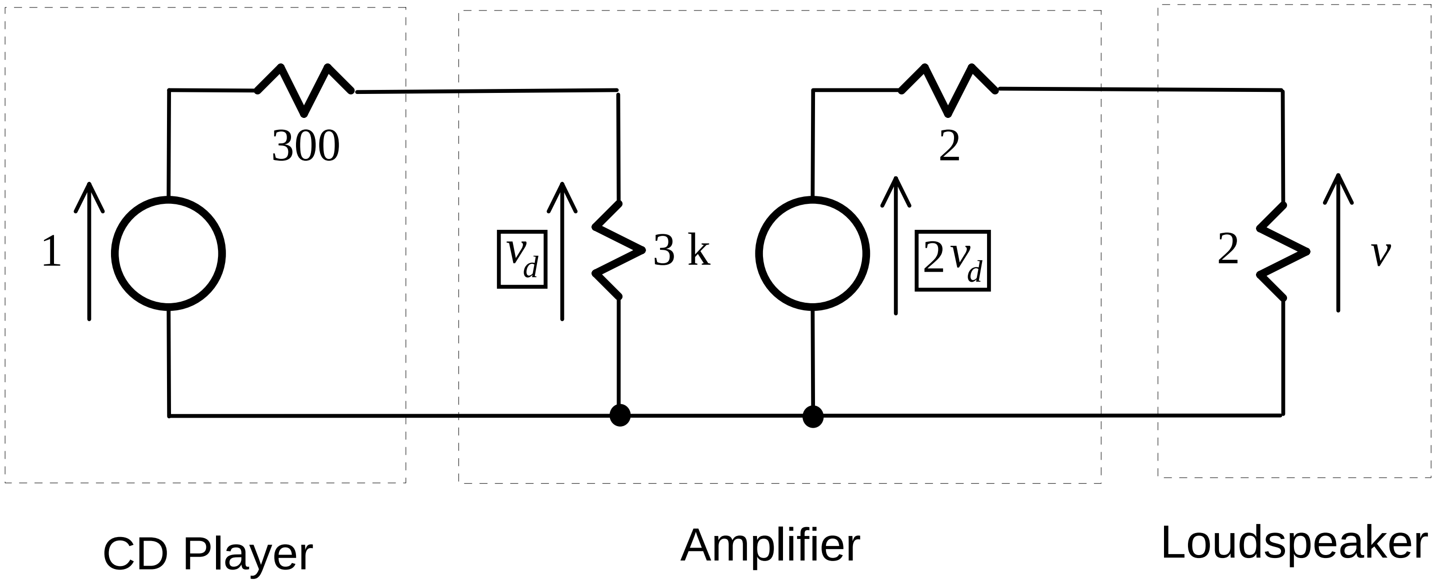
Superposition: Applicable only to linear systems

2.22 Dependent Source

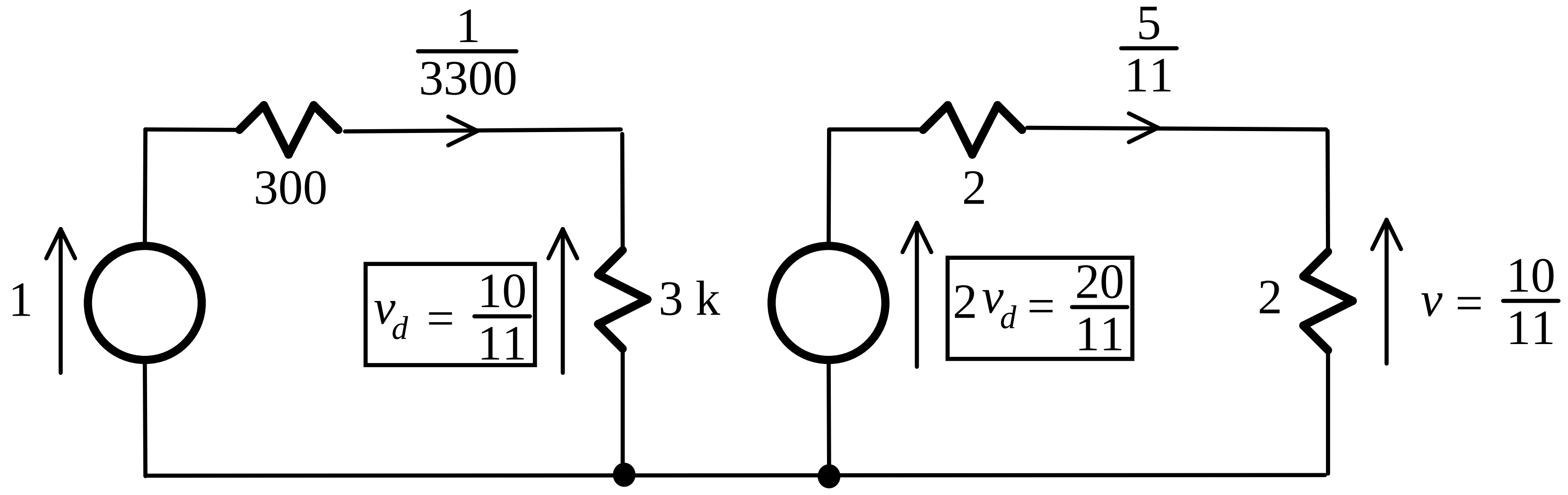
Consider the following system:



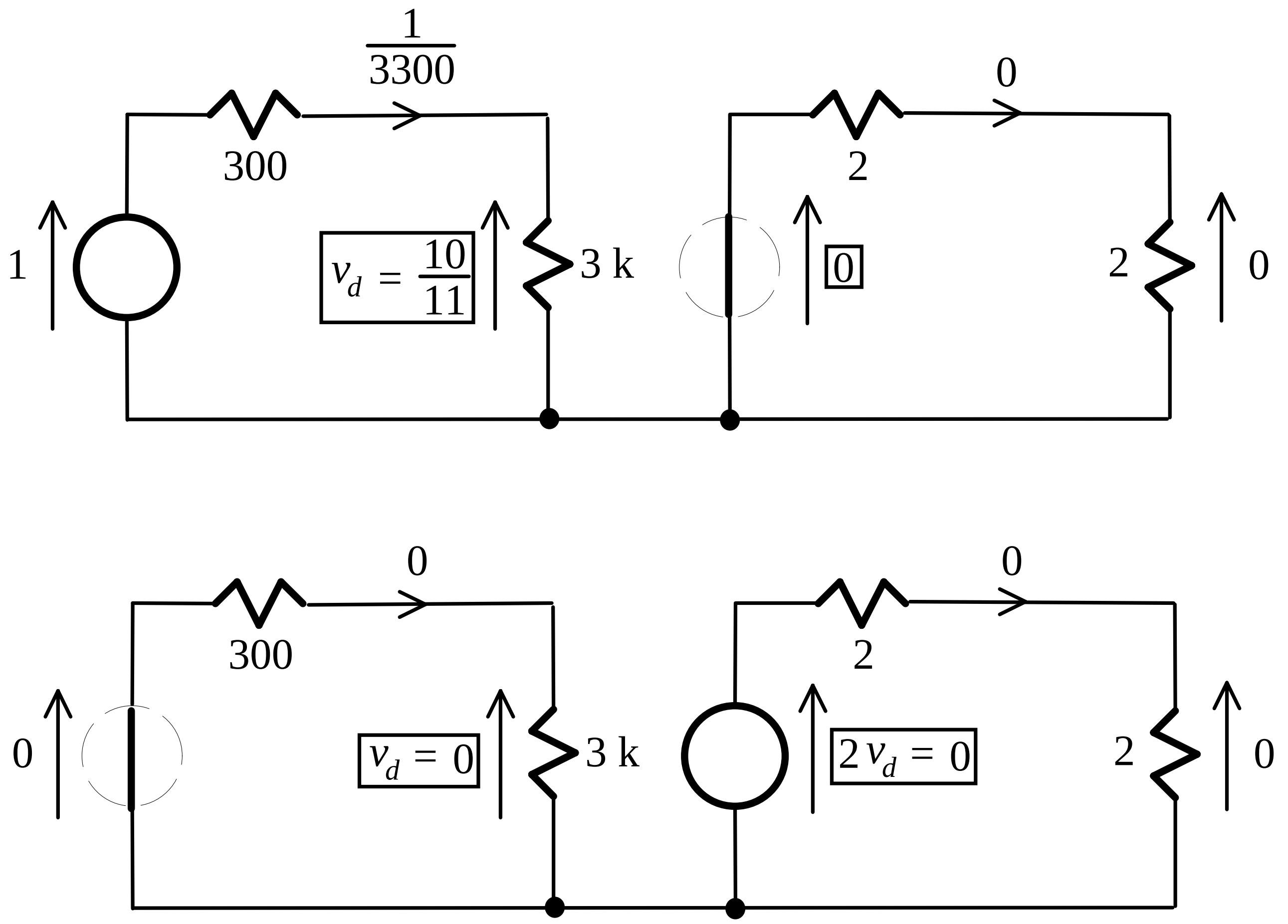
This may be represented by



Note that the source in the Amplifier block is a **dependent source**. Its value depends on v_d , the voltage across the inputs of the amplifier. Using KCL and KVL, the voltage v can be easily found:



However, if we use the principle of superposition treating the dependent source as an independent source (which is wrong!), the value of v will be 0.

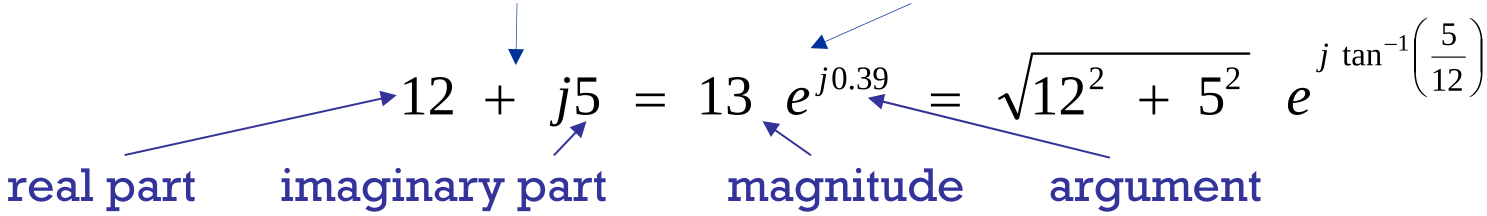


Dependent sources, which depend on other voltages/currents in the circuit and are therefore not independent excitations, cannot be removed when the principle of superposition is used. They should be treated like other passive components such as resistors in circuit analysis.

3. AC Circuit Analysis

Appendix Materials: Operations of Complex Numbers

Coordinates: Cartesian Coordinate and Polar Coordinate



$$12 + j5 = 13\, e^{j0.39} = \sqrt{12^2 + 5^2}\; e^{j\tan^{-1}\left(\frac{5}{12}\right)}$$

real part

imaginary part

magnitude

argument

Euler's Formula:

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

Additions: It is easy to do additions (subtractions) in Cartesian coordinate.

$$\boxed{(a + jb) + (v + jw) = (a + v) + j(b + w)}$$

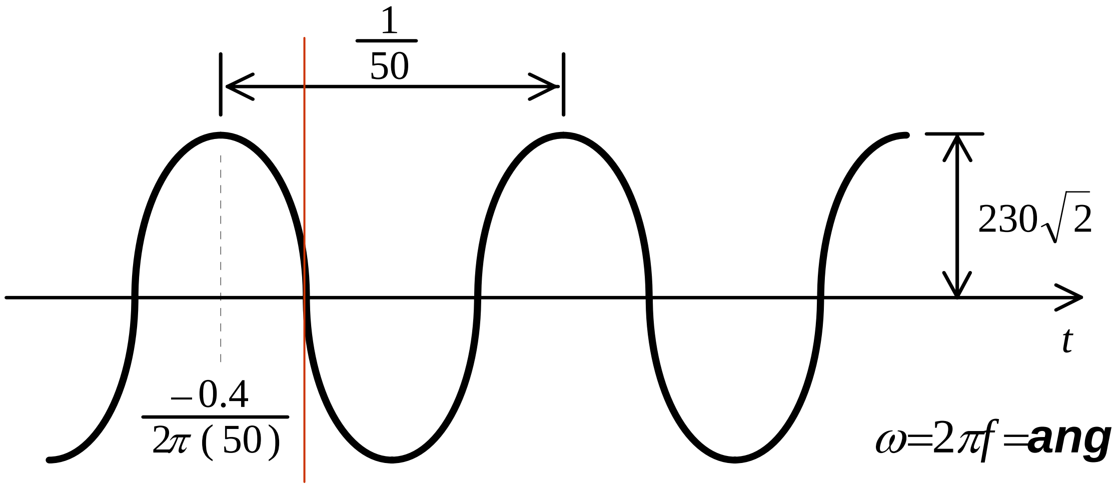
Multiplication: It is easy to do multiplication (division) in Polar coordinate.

$$\boxed{re^{j\theta} \cdot ue^{j\omega} = (ru)e^{j(\theta+\omega)}} \qquad \boxed{\frac{re^{j\theta}}{ue^{j\omega}} = \frac{r}{u}\, e^{j(\theta-\omega)}}$$

3.1 AC Sources

Voltages and currents in DC circuit are constants and do not change with time.

In AC (**alternating current**) circuits, voltages and currents change with time in a sinusoidal manner. The most common ac voltage source is the mains:



$\theta =$ ***phase*** $= 0.4\text{rad}$

$f =$ ***frequency*** $= 50\text{Hz}$

$\omega = 2\pi f =$ ***angular frequency*** $= 100\pi = 314\text{rad/s}$

$T = \frac{1}{f} =$ ***period*** $= \frac{1}{50} = 0.02\text{s}$

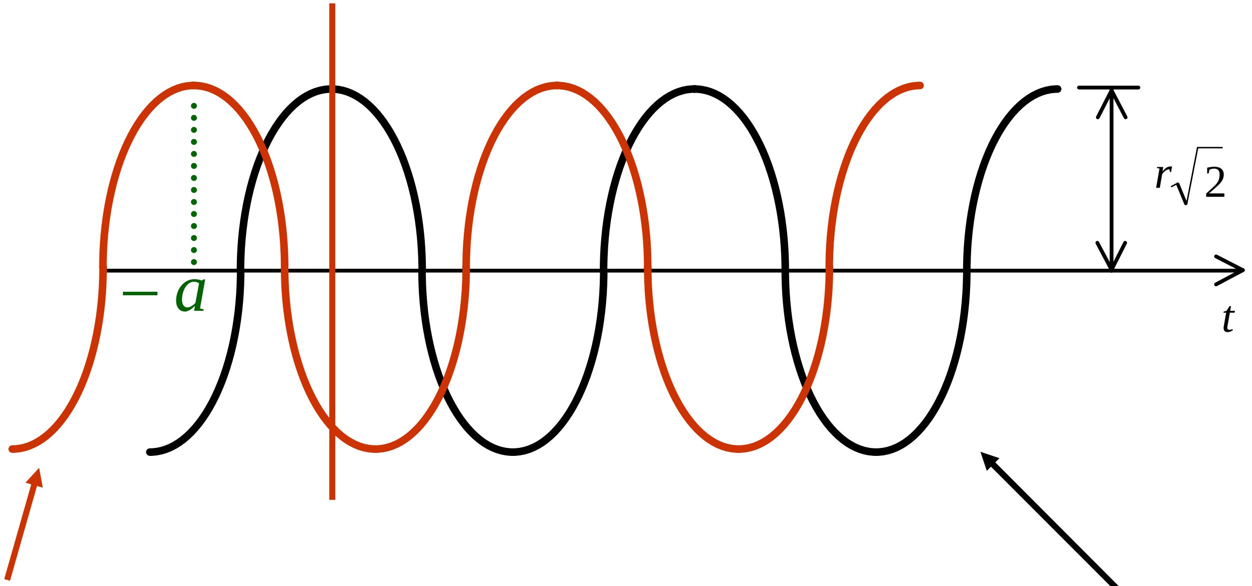
$$v(t) = \sqrt{2}r\cos(2\pi ft+\theta) = \sqrt{2}r\cos(\omega t+\theta) = \sqrt{2}r\cos\left(\frac{2\pi t}{T}+\theta\right)$$

$$= 230\sqrt{2}\cos(100\pi t+0.4)$$

$\sqrt{2}r =$ ***peak value*** $= 230\sqrt{2} = 324\text{V}$

$r =$ ***rms (root mean square) value*** $= 230\text{V}$

How to find the phase for a sinusoidal function?



$$v(t) = r\sqrt{2}\cos(\omega[t + a]) = r\sqrt{2}\cos(\omega t + \omega a)$$

$$v(t) = r\sqrt{2}\cos(\omega t)$$

$$\Rightarrow \boxed{\text{Phase } \theta = \omega a}$$

$$-\pi \le \theta \le \pi$$

$$\theta = \omega a = 314\left(\frac{0.4}{2\pi(50)}\right) = 0.4$$

for previous example.

3.2 Phasor

Euler's Formula: $e^{j\omega} = \cos(\omega) + j\sin(\omega)$

A sinusoidal voltage/current is represented using complex number format:

$$v(t) = \sqrt{2}r\cos(\omega t+\theta) = \sqrt{2}r\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

The advantage of this can be seen if, say, we have to add 2 sinusoidal voltages given by:

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) \qquad v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] \qquad v_2(t) = 5\sqrt{2}\cos\left(\omega - \frac{\pi}{4}\right) = \mathrm{Re}\left[\left(5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$v_1(t) + v_2(t) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{\omega t}\right)\right] = \mathrm{Re}\left[\left(6.47e^{-j0.32}\right)\left(\sqrt{2}e^{j\omega t}\right)\right] = 6.47\sqrt{2}\cos(\omega t - 0.32)$$

Note that the complex time factor $\sqrt{2}e^{j\omega t}$ appears in all the expressions. If we represent $v_1(t)$ and $v_2(t)$ by the complex numbers or ***phasors***:

$$V_1 = 3e^{j\frac{\pi}{6}} \text{ representing } v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right)$$

$$V_2 = 5e^{-j\frac{\pi}{4}} \text{ representing } v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

then the phasor representation for $v_1(t) + v_2(t)$ will be

$$V_1 + V_2 = 3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}} = 6.47e^{-j0.32} \quad \text{representing } v_1(t) + v_2(t) = 6.47\sqrt{2}\cos(\omega t - 0.32)$$

$$3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}} = 3\left(\cos\left(\frac{\pi}{6}\right) + j\sin\left(\frac{\pi}{6}\right)\right) + 5\left(\cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right)\right) = 6.14 - j2.03 = 6.47e^{-j0.32}$$

Euler's Formula: $e^{j\omega} = \cos(\omega) + j\sin(\omega)$

By using phasors, a time-varying ac voltage

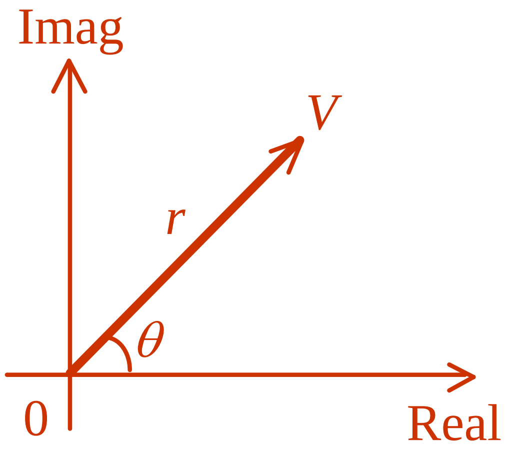
$$v(t)=\sqrt{2}r\cos(\omega t+\theta)=\mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

becomes a simple complex time-invariant number/voltage $V=re^{j\theta}=r\angle\theta$

$r=|V|=$ magnitude/modulus of V = r.m.s. value of $v(t)$

$\theta=\mathrm{Arg}[V]=$ phase of V

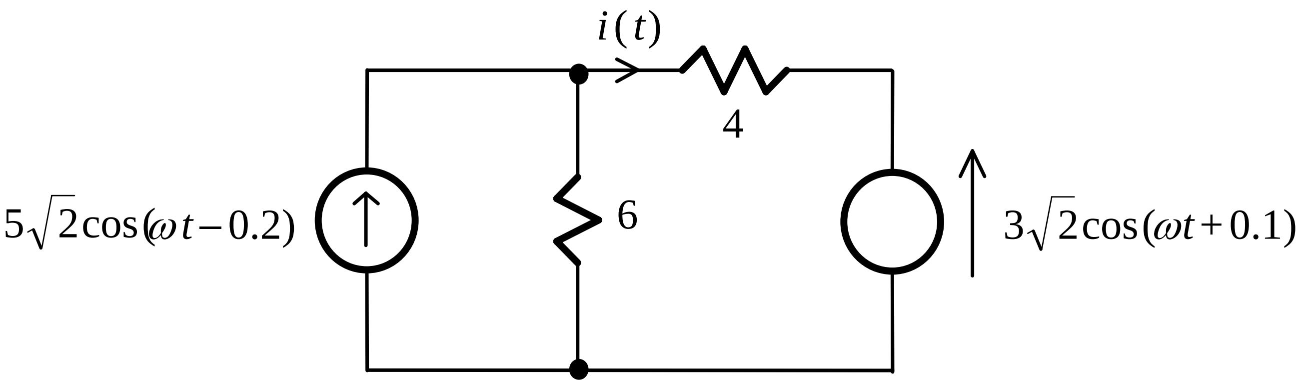
Graphically, on a phasor diagram:

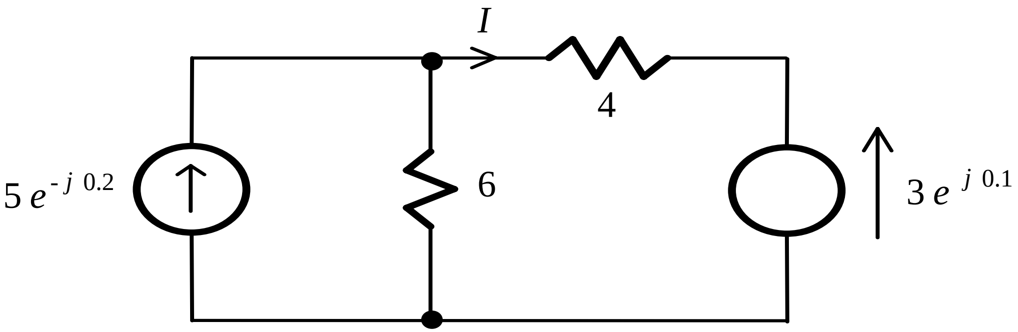


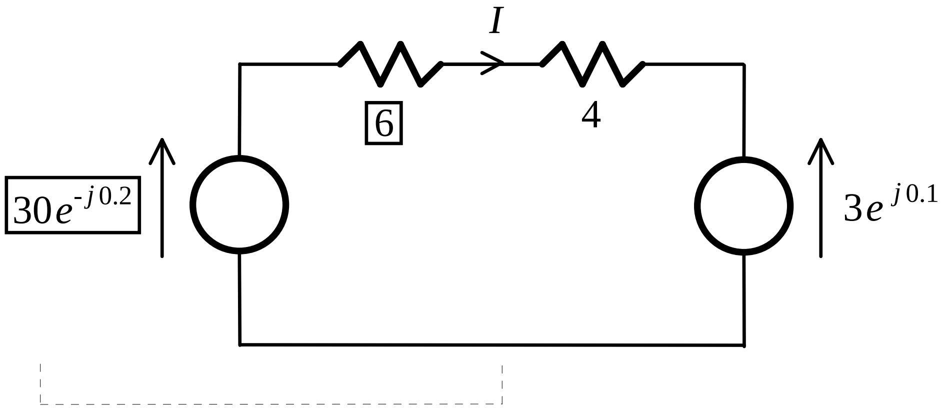
Complex Plane

Using phasors, all time-varying ac quantities become complex dc quantities and all dc circuit analysis techniques can be employed for ac circuit with virtually no modification.

Example:

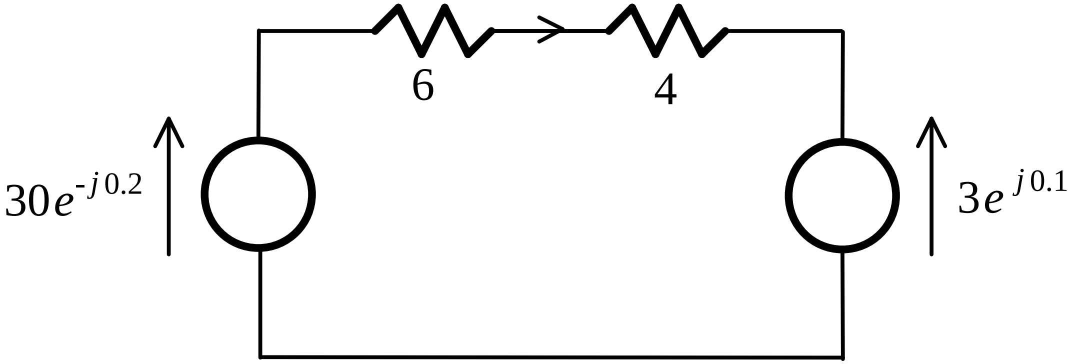






Thevenin's equivalent circuit
for current soure and 6Ω resistor

$$\boxed{I = \frac{30e^{-j0.2} - 3e^{j0.1}}{10} = 2.64 - j0.63 = 2.71e^{-j0.23}}$$



$$\begin{aligned} I &= \frac{30e^{-j0.2} - 3e^{j0.1}}{10} \\ &= 3\left[\cos(-0.2) + j\sin(-0.2)\right] - 0.3\left[\cos 0.1 + j\sin 0.1\right] \\ &= (2.940 - j0.596) - (0.299 + j0.030) = 2.641 - j0.626 \\ &= \sqrt{2.641^2 + 0.626^2}\, e^{j\tan^{-1}\left(\frac{-0.626}{2.641}\right)} = 2.71e^{-j0.23} \end{aligned}$$

$$\Rightarrow i(t) = 2.71\sqrt{2}\cos(\omega t - 0.23)$$

3.3 Root Mean Square (rms) Value

For the ac voltage

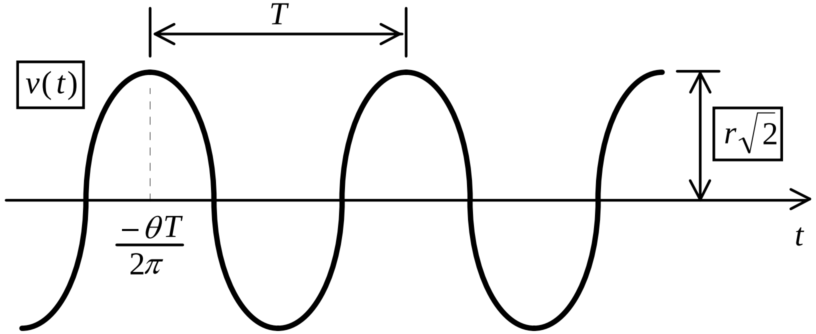
$$v(t) = \sqrt{2}r\cos(2\pi f t + \theta) = \sqrt{2}r\cos\left(\frac{2\pi t}{T} + \theta\right)$$

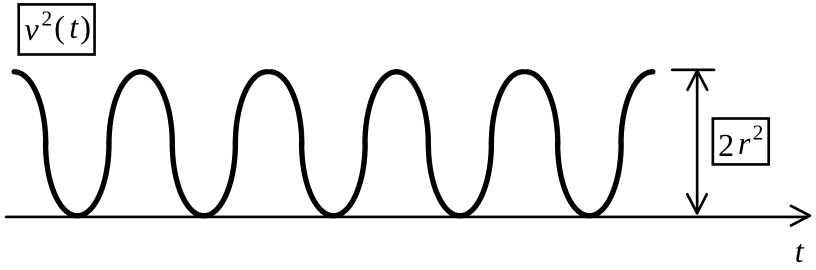
$$\downarrow$$

$$\boxed{\cos(2x) = 2\cos^2(x) - 1}$$

$$\downarrow$$

$$v^2(t) = 2r^2\cos^2\left(\frac{2\pi t}{T} + \theta\right) = r^2\left[1 + \cos\left(\frac{4\pi t}{T} + 2\theta\right)\right]$$





The average or mean of the square value is

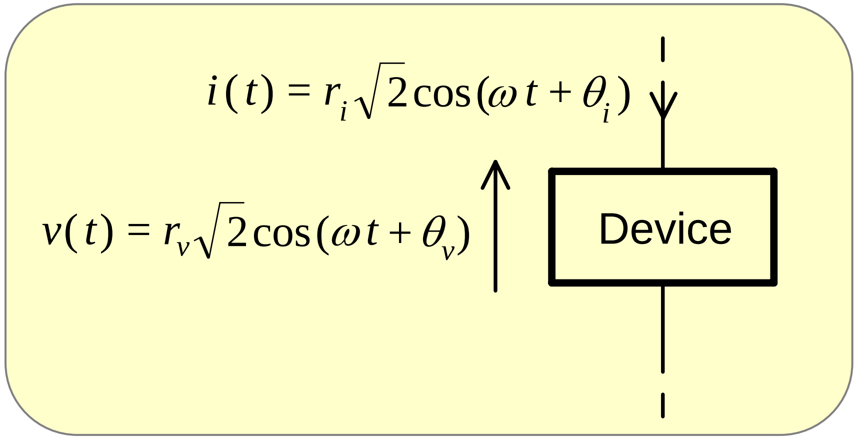
$$\frac{1}{1\,\text{period}}\int_{1\,\text{period}} v^2(t)\,dt = \frac{1}{T}\int_0^T v^2(t)dt = \frac{1}{T}\int_0^T r^2\left[1 + \cos\left(\frac{4\pi t}{T} + 2\theta\right)\right]dt = \frac{1}{T}\int_0^T r^2 dt = r^2$$

The square root of this or the ***rms*** value of $v(t)$ is the rms value of $\sqrt{2}r\cos(\omega t + \theta) = r$

Side Note: rms value can be defined for any periodical signal.

3.4 Power

Consider the ac device:



$$i(t) = r_i \sqrt{2} \cos(\omega t + \theta_i)$$

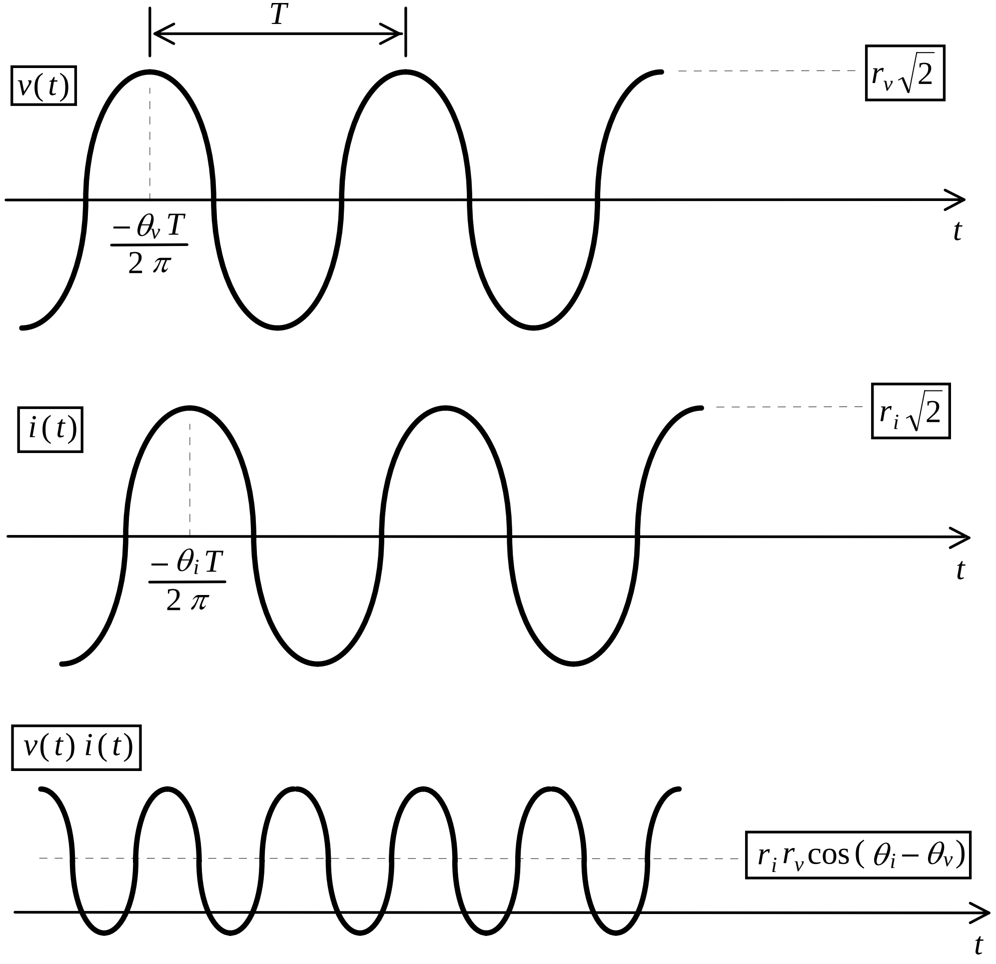
$$v(t) = r_v \sqrt{2} \cos(\omega t + \theta_v)$$

Using $2\cos(x_1)\cos(x_2) = \cos(x_1 - x_2) + \cos(x_1 + x_2)$, the ***instantaneous power*** consumed is

$$p(t) = i(t)v(t) = 2 r_i r_v \cos(\omega t + \theta_i)\cos(\omega t + \theta_v) = r_i r_v [\cos(\theta_i - \theta_v) + \cos(2\omega t + \theta_i + \theta_v)]$$

The ***average power*** consumed is

$$p_{av} = \frac{1}{1\,\text{period}} \int_{1\,\text{period}} p(t)dt = \frac{r_i r_v}{T} \int_0^T \left[\cos(\theta_i - \theta_v) + \cos\left(\frac{4\pi t}{T} + \theta_i + \theta_v \right) \right] dt = r_i r_v \cos(\theta_i - \theta_v)$$



In phasor notation:

$$V = r_v e^{j\theta_v}, \qquad V^* = r_v e^{-j\theta_v}$$

$$I = r_i e^{j\theta_i}, \qquad I^* = r_i e^{-j\theta_i}$$

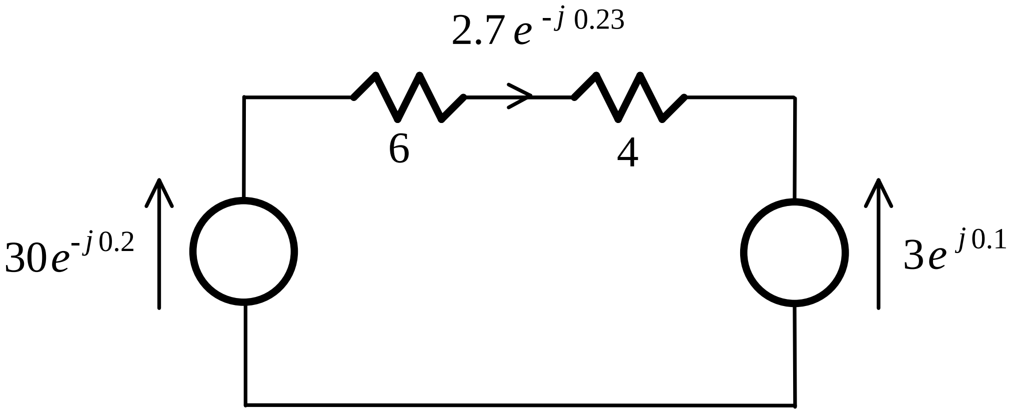
$$V^* I = r_v r_i e^{j(\theta_i - \theta_v)}$$

$$V I^* = r_v r_i e^{j(\theta_v - \theta_i)}$$

$$p_{av} = r_v r_i \cos(\theta_v - \theta_i) = r_v r_i \cos(\theta_i - \theta_v) = \mathrm{Re}\left[r_v r_i e^{j(\theta_v - \theta_i)}\right] = \mathrm{Re}\left[V^* I\right] = \mathrm{Re}\left[V I^*\right]$$

Note that the formula $p_{av} = \mathrm{Re}[I^* V]$ is based on rms voltages and currents. Also, this is valid for dc circuits, which is a special case of ac circuits with $f = 0$ and V and I having real values.

Example: Consider the ac circuit,



$$I = \frac{30e^{-j0.2} - 3e^{j0.1}}{4+6} = 2.7148e^{-j0.2327}$$

$$3e^{j0.1}\text{source}: \ \mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^*\left(3e^{j0.1}\right)\right] = \mathrm{Re}\left[8.1e^{j0.33}\right] = 8.1\cos(0.33) = 7.66$$

$$30e^{-j0.2}\text{source}: \ \mathrm{Re}\left[-\left(2.7e^{-j0.23}\right)^*\left(30e^{-j0.2}\right)\right] = -81\cos(0.03) = -80.96$$

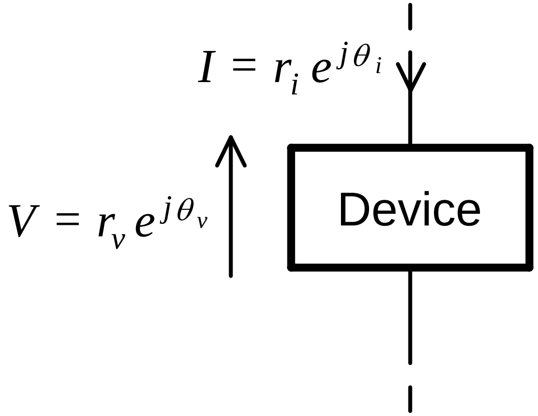
$$6\Omega\,\text{resistor}: \ \mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^*\left(6\times 2.7e^{-j0.23}\right)\right] = 6(2.7)^2 = 43.74$$

$$4\Omega\,\text{resistor}: \ \mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^*\left(4\times 2.7e^{-j0.23}\right)\right] = 4(2.7)^2 = 29.16$$

$$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} = 0$$

3.5 Power Factor

Consider the ac device:



Ignoring phase difference between V and I, the voltage-current rating or ***apparent power*** consumed is

$$\text{Apparent power} = \text{voltage - current rating} = |V||I| = r_v r_i \text{ VA}$$

However, the actual power consumed is

$$\text{Actual power} = \mathrm{Re}\left[V^* I\right] = r_v r_i \cos(\theta_i - \theta_v) \text{W}$$

The ratio of the these powers is the ***power factor*** of the device:

$$\text{Power factor} = \frac{\text{Actual power}}{\text{Apparent power}} = \cos(\theta_i - \theta_v)$$

This has a maximum value of 1 when $\text{Unity power factor} \Leftrightarrow I \text{ and } V \text{ in phase} \Leftrightarrow \theta_i = \theta_v$

The power factor is said to be ***leading*** or ***lagging*** if

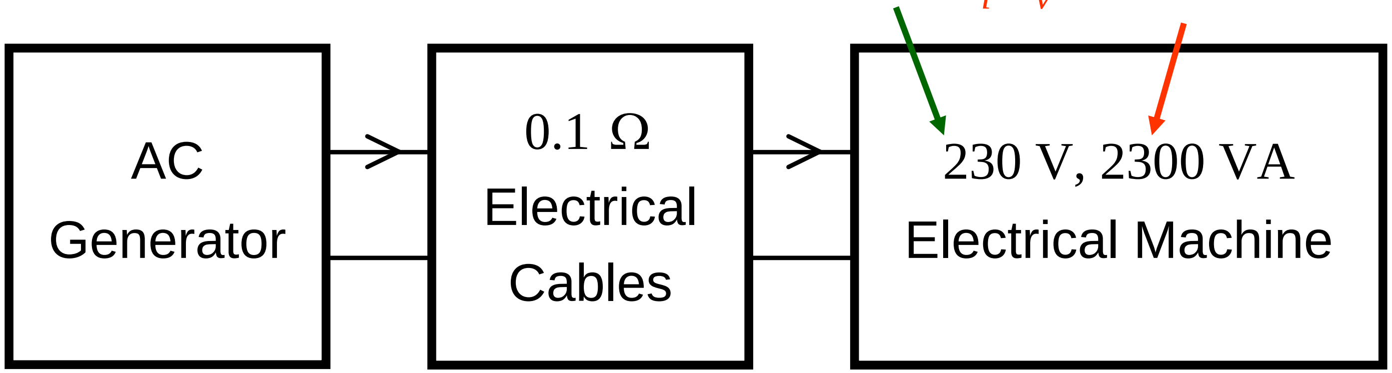
$$\text{Leading power factor} \Leftrightarrow I \text{ leads } V \text{ in phase} \Leftrightarrow \theta_i > \theta_v$$

$$\text{Lagging power factor} \Leftrightarrow I \text{ lags } V \text{ in phase} \Leftrightarrow \theta_i < \theta_v$$

Consider the following ac system:

$r_v = 230$

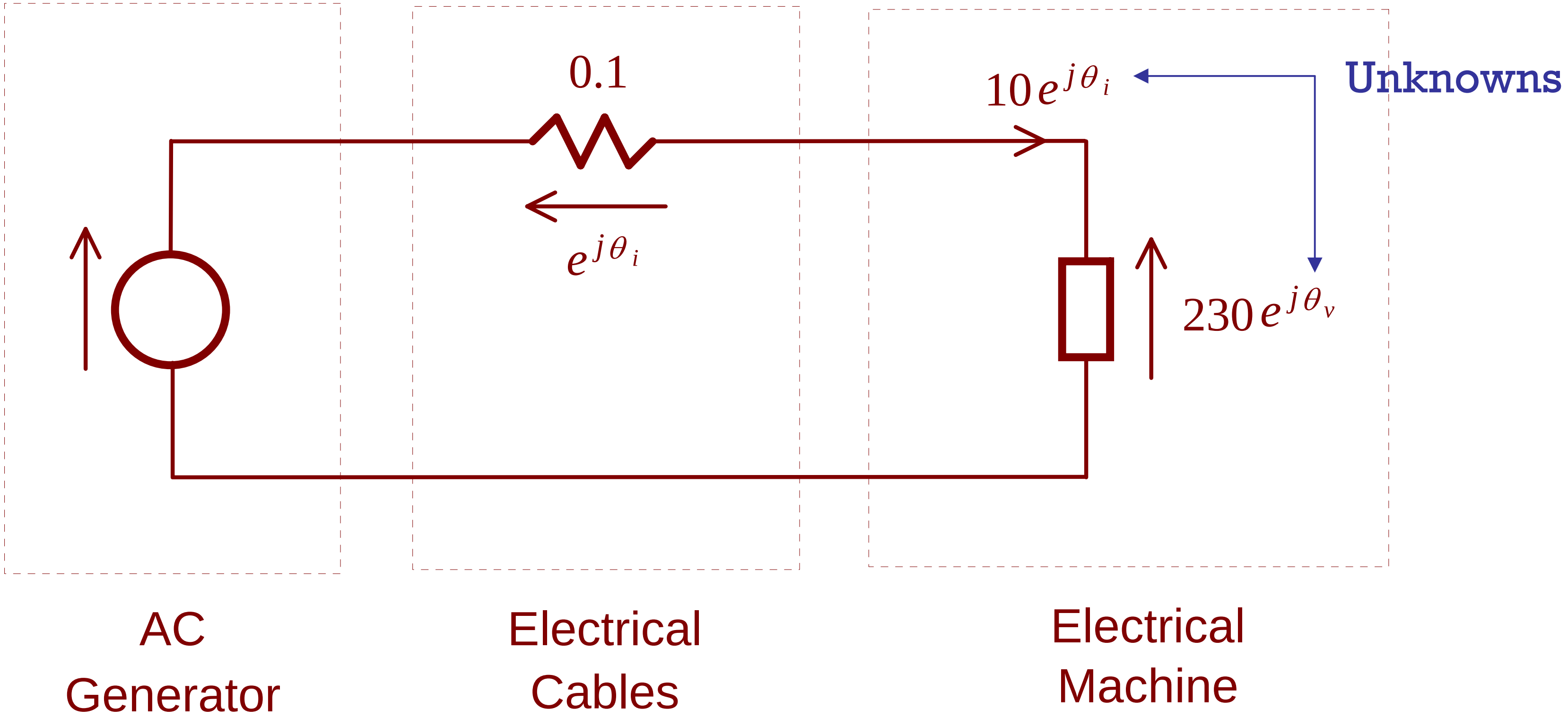
$r_i \, r_v = 2300 \Rightarrow r_i = 10$



AC Generator

0.1 Ω Electrical Cables

230 V, 2300 VA Electrical Machine

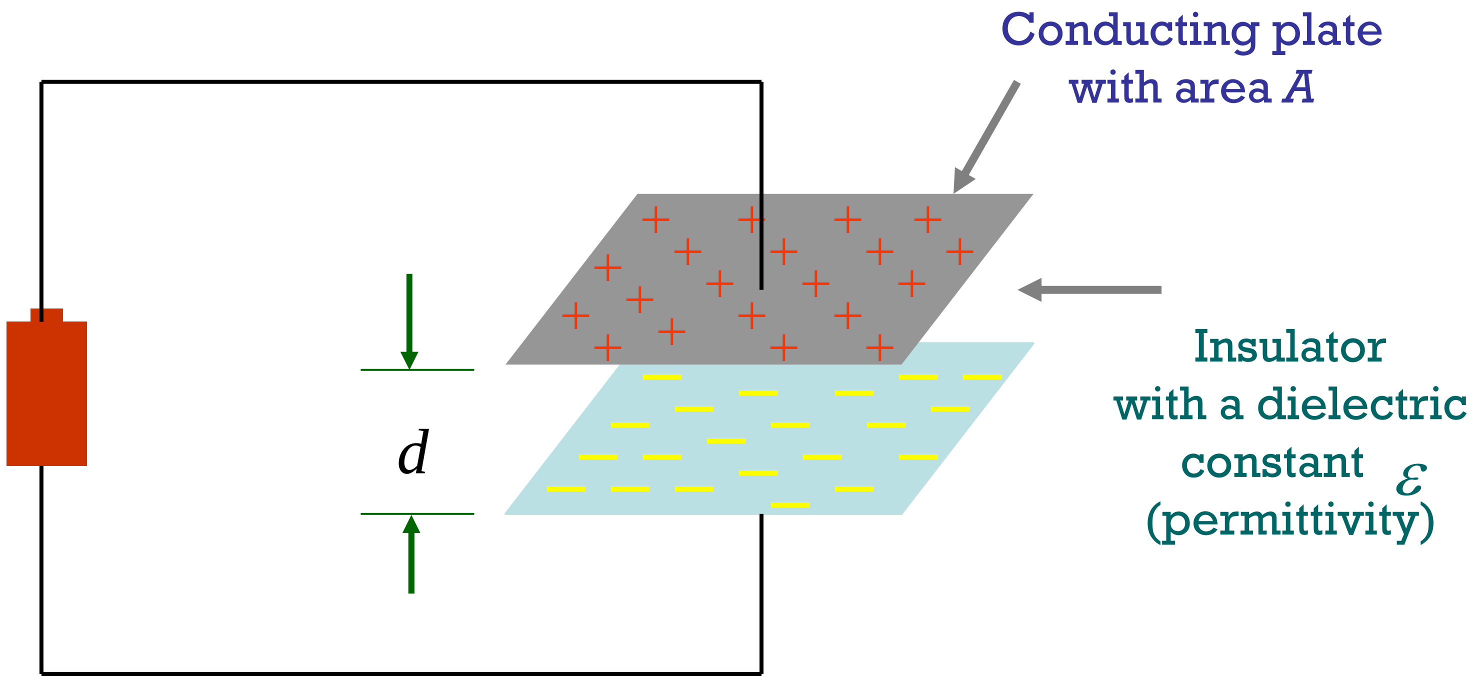


Power consumed by the machine and power loss at different power factors are:

Voltage-current rating	$2300\ \text{VA}$	$2300\ \text{VA}$	$2300\ \text{VA}$
Voltage across machine	$230\ \text{V}$	$230\ \text{V}$	$230\ \text{V}$
Current	$10\ \text{A}$	$10\ \text{A}$	$10\ \text{A}$
Power factor	0.11 leading	1	0.11 lagging
$\theta_i - \theta_v$	$\cos^{-1}(0.11) = 1.4\text{rad}$	0	$-\cos^{-1}(0.11) = -1.46\text{rad}$
Power consumed by machine	$(2300)(0.11) = 253\text{W}$	$(2300)(1) = 2300\text{W}$	$(2300)(0.11) = 253\text{W}$
Power loss in cables	$(0.1)(10)^2 = 10\text{W}$	$(0.1)(10)^2 = 10\text{W}$	$(0.1)(10)^2 = 10\text{W}$

3.6 Capacitor

A ***capacitor*** consists of parallel metal plates for storing electric charges.

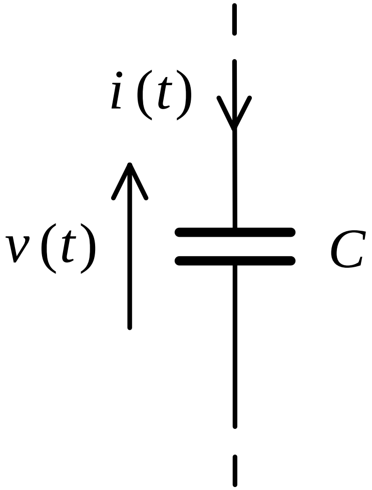


The capacitance of the capacitor is given by $C = \varepsilon \dfrac{A}{d}$ F or Farad

Area of metal plates required to produce a 1F capacitor in the free space if *d* = 0.1 mm is

$$A = \frac{Cd}{\varepsilon} = \frac{1\text{F} \times 0.0001\text{m}}{8.85 \times 10^{-12}\ \text{F/m}} = 11.3\ (\text{km})^2$$

The circuit symbol for an ideal capacitor is:



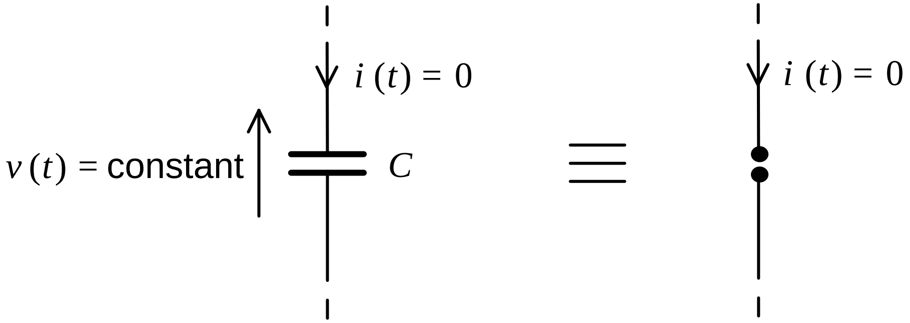
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$i(t) = C\frac{dv(t)}{dt}$$

For dc circuits:

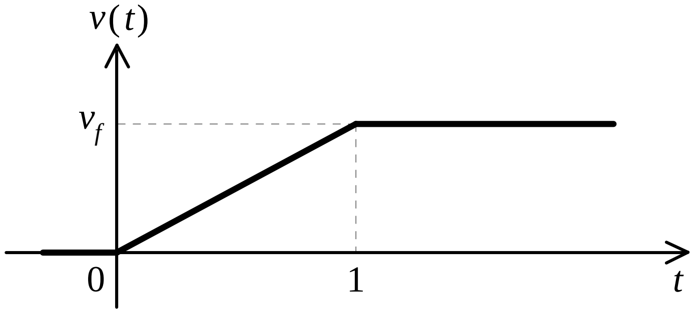
$$v(t) = \text{constant} \Rightarrow \frac{dv(t)}{dt} = 0 \quad \Rightarrow \quad i(t) = 0$$

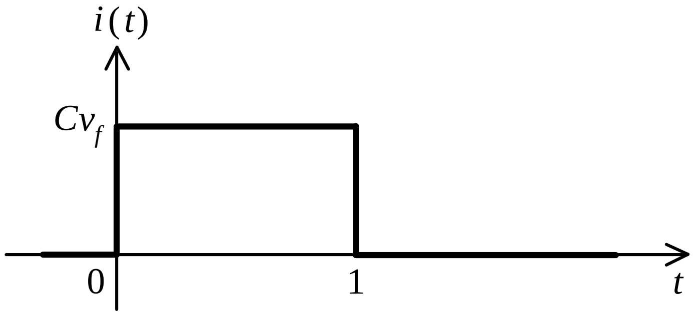
and the capacitor is equivalent to an open circuit:



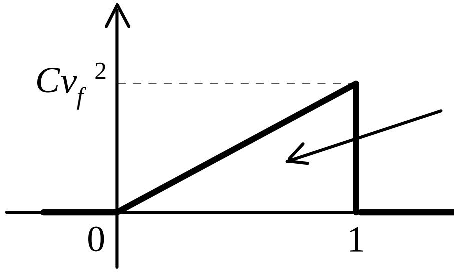
This is why we don't consider the capacitor in DC circuits.

Consider the change in voltage, current and power supplied to the capacitor as indicated below:





$p(t) = v(t)\, i(t) =$ Instantaneous power **consumed**



Area = Energy stored = $\dfrac{Cv_f^{\,2}}{2}$

In general, the total **energy stored** in the electric field established by the charges on the capacitor plates at time is

$$e(t) = \frac{Cv^2(t)}{2}$$

Proof.

$$e(t) = \int_{-\infty}^{t} p(x)dx = \int_{-\infty}^{t} v(x)i(x)dx$$

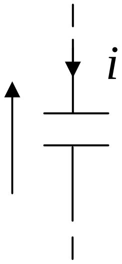
$$= \int_{-\infty}^{t} v(x)C\frac{dv(x)}{dx}dx$$

$$= C\int_{-\infty}^{t} v(x)dv(x) = \frac{C}{2}v^2(x)\Big|_{-\infty}^{t}$$

$$= \frac{C}{2}\left[v^2(t) - v^2(-\infty)\right]$$

$$= \frac{Cv^2(t)}{2}, \qquad \text{if} \quad v(-\infty) = 0.$$

Now consider the operation of a capacitor in an ac circuit:

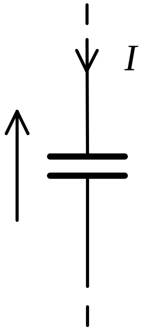


$$v(t) = r_v\sqrt{2}\cos(\omega t+\theta_v)$$

$$i(t) = C\frac{dv(t)}{dt} = -\omega C r_v\sqrt{2}\sin(\omega t+\theta_v)$$

$$= \omega C r_v\sqrt{2}\cos(\omega t+\theta_v+\frac{\pi}{2})$$

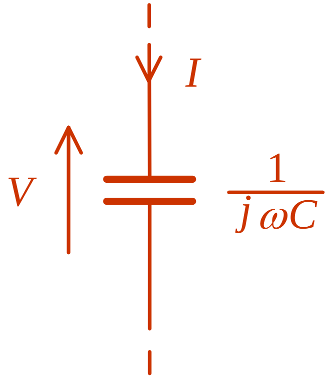
In phasor format:



$$I = \omega C r_v e^{j\theta_v} e^{j\frac{\pi}{2}} = j\omega C r_v e^{j\theta_v} = j\omega C V$$

$$V = r_v e^{j\theta_v}$$

$$\Longrightarrow \quad \frac{V}{I} = \frac{1}{j\omega\ C} \quad \Longrightarrow$$



With phasor representation, the capacitor behaves as if it is a resistor with a "complex resistance" or an **impedance** of

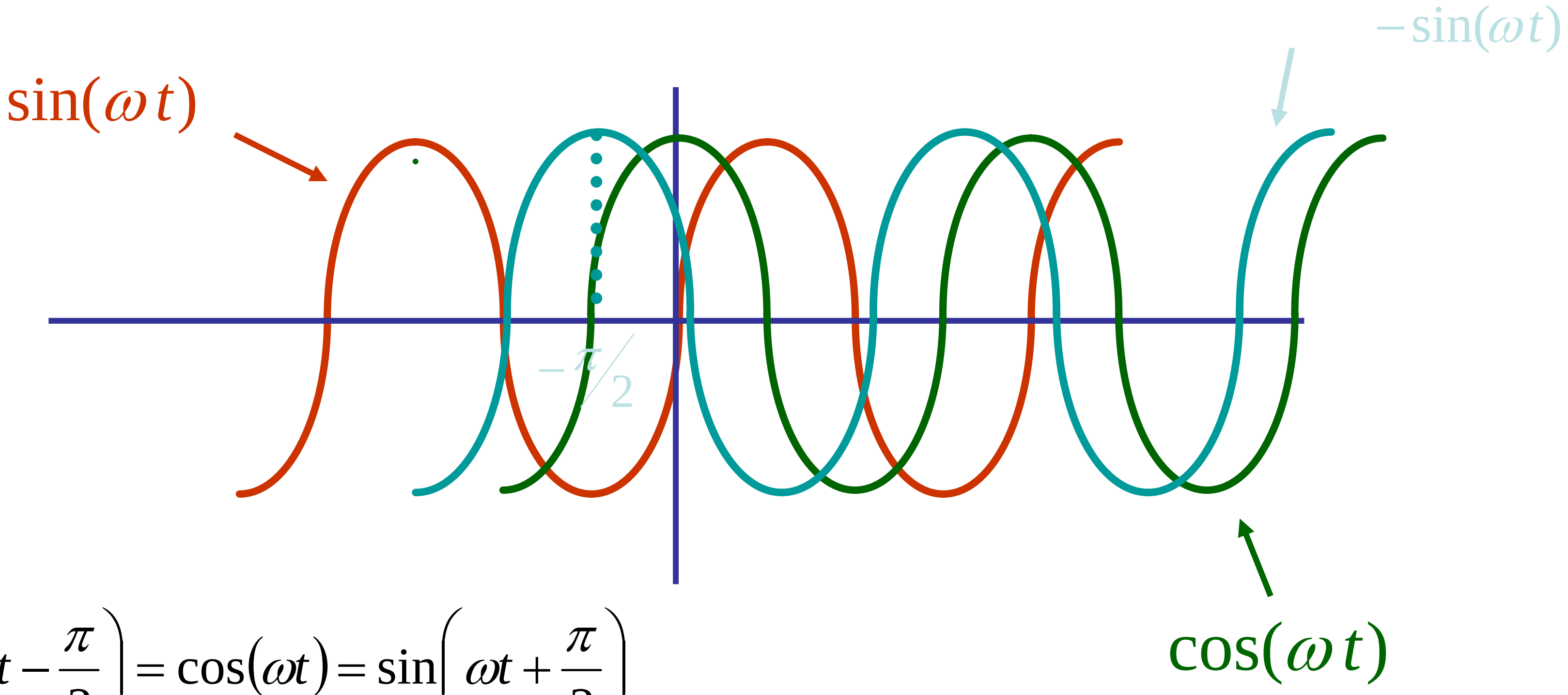
$$Z_C = \frac{1}{j\omega\ C} \quad \Longrightarrow \quad p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* I Z_C\right] = \mathrm{Re}\left[\frac{|I|^2}{j\omega\ C}\right] = 0$$

An ideal capacitor is a non-dissipative but energy-storing device.

Since the phase of *I* relative to *V* that of is

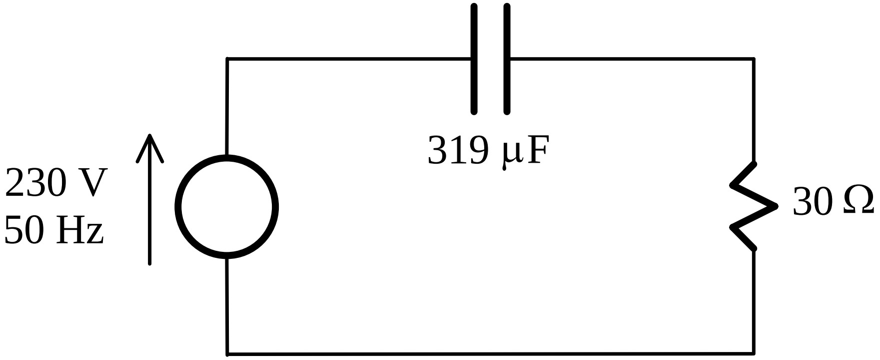
$$\mathrm{Arg}[I]-\mathrm{Arg}[V]=\mathrm{Arg}\left[\frac{I}{V}\right]=\mathrm{Arg}\left[\frac{1}{Z_C}\right]=\mathrm{Arg}[j\omega\ \ C]=90^0$$

the ac current $i(t)$ of the capacitor leads the voltage $v(t)$ by 90°.

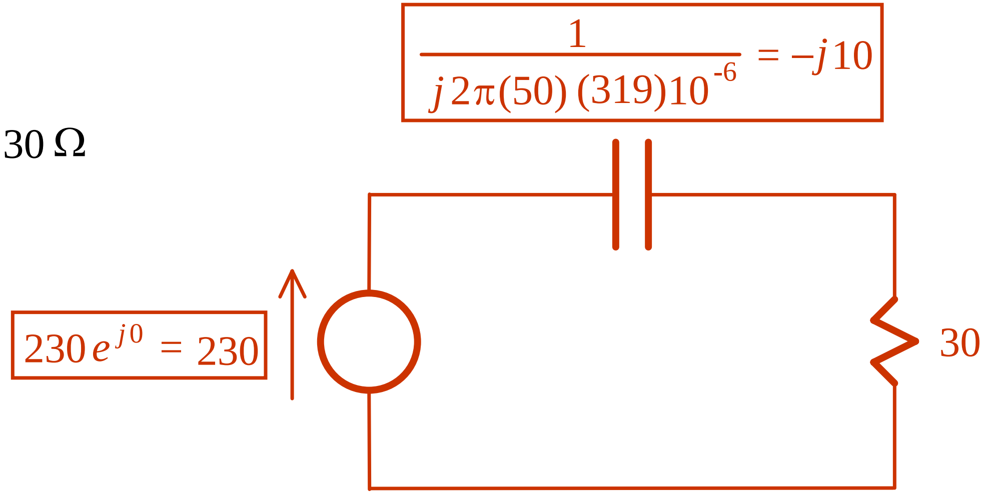


$$-\sin\left(\omega t-\frac{\pi}{2}\right)=\cos(\omega t)=\sin\left(\omega t+\frac{\pi}{2}\right)$$

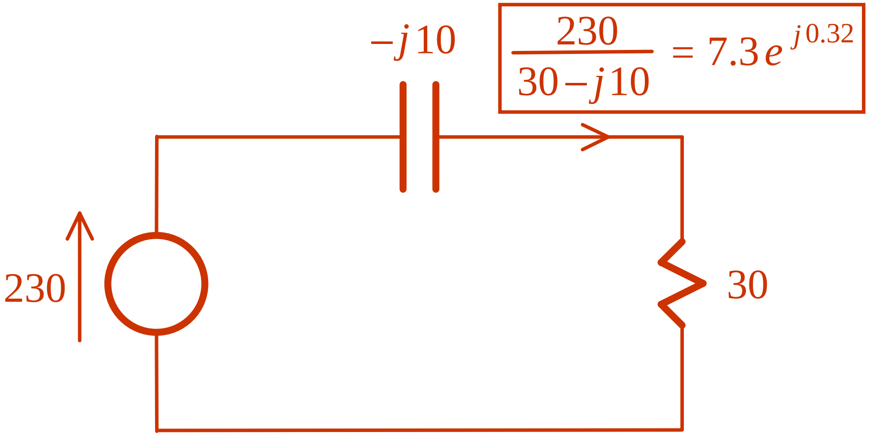
Example: Consider the following ac circuit:



In phasor notation (taking the source to have a reference phase of 0):


$$\boxed{\frac{1}{j2\pi(50)\,(319)10^{-6}} = -j10}$$

$$\boxed{230e^{j0} = 230}$$


$-j10$

$$\boxed{\frac{230}{30-j10} = 7.3e^{j0.32}}$$

Total circuit impedance	$Z = (30 - j10)\,\Omega$
Total circuit *reactance*	$X = \mathrm{Im}[Z] = \mathrm{Im}[30 - j10] = -10\,\Omega$
Total circuit resistance	$R = \mathrm{Re}[Z] = \mathrm{Re}[30 - j10] = 30\,\Omega$
Current (rms)	$\lvert I \rvert = 7.3\mathrm{A}$
Current (peak)	$\sqrt{2}\lvert I \rvert = 7.3\sqrt{2} = 10\mathrm{A}$
Source *V*-*I* phase relationship	I leads by 0.32rad
Power factor of entire circuit	$\cos(0.32) = 0.95$ leading
Power supplied by source	$\mathrm{Re}\left[(230)^{*}\left(7.3e^{j0.32}\right)\right] = (230)(7.3)\cos(0.32) = 1.6\ \mathrm{kW}$
Power consumed by resistor	$\mathrm{Re}\left[\left(7.3e^{j0.32}\right)^{*}\left(30 \times 7.3e^{j0.32}\right)\right] = (7.3)^{2}30 = 1.6\ \mathrm{kW}$

Impedance, Resistance, Reactance,

Admittance, Conductance, and Susceptance

Relations?

Impedance:

$$Z = R + jX$$

Resistance → R, Reactance → X

Admittance:

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{(R + jX)(R - jX)}$$

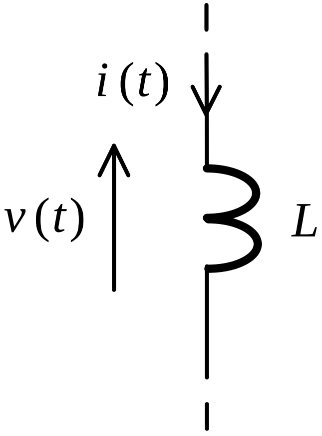
$$= \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} + j\frac{-X}{R^2 + X^2}$$

$$= G + jB$$

Conductance → G, Susceptance → B

3.7 Inductor

An **inductor** consists of a coil of wires for establishing a magnetic field. The circuit symbol for an ideal inductor is:



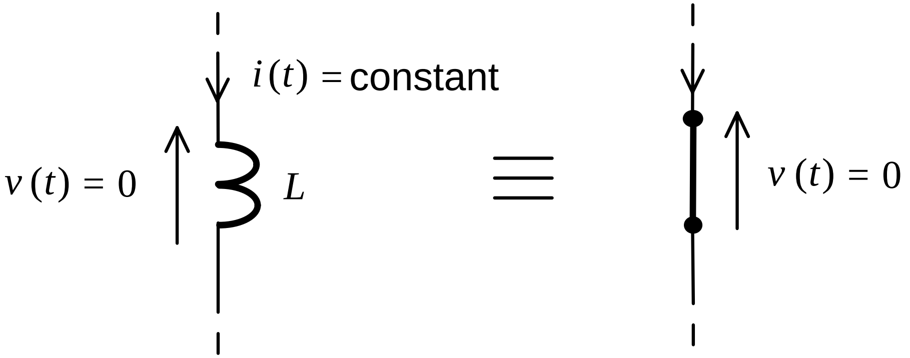
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$\boxed{v(t) = L\frac{di(t)}{dt}}$$

For dc circuits:

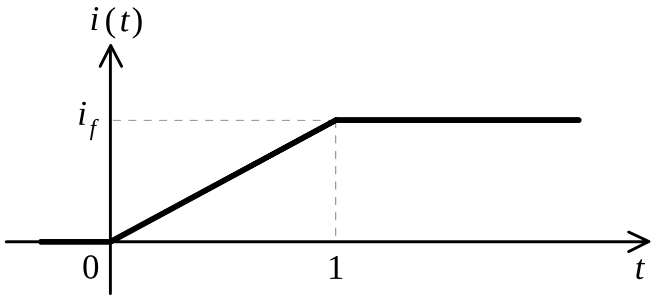
$$i(t) = \text{constant} \Rightarrow \frac{di(t)}{dt} = 0 \Rightarrow v(t) = 0$$

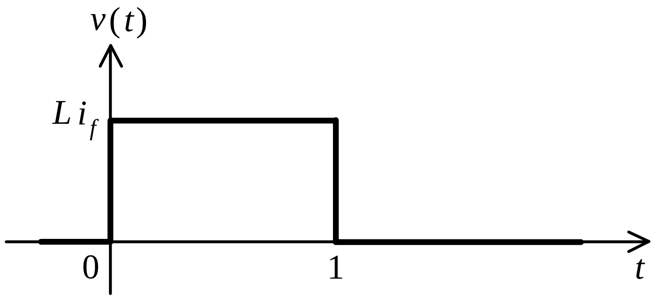
and the inductor is equivalent to a short circuit:



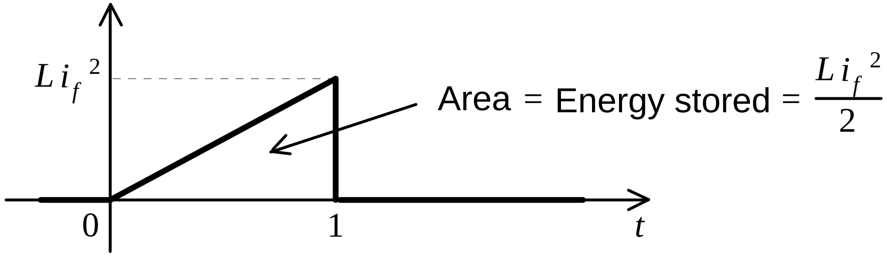
That is why there is nothing interesting about the inductor in DC circuits.

Consider the change in voltage, current and power supplied to the inductor as indicated below:





$p(t) = v(t)\,i(t)$ =Instantaneous power consumed



Area = Energy stored = $\dfrac{Li_f^{\,2}}{2}$

In general, the total energy stored in the magnetic field established by the current $i(t)$ in the inductor at time t is given by

$$e(t) = \frac{Li^2(t)}{2}$$

$$e(t) = \int_{-\infty}^{t} p(x)dx = \int_{-\infty}^{t} v(x)i(x)dx$$

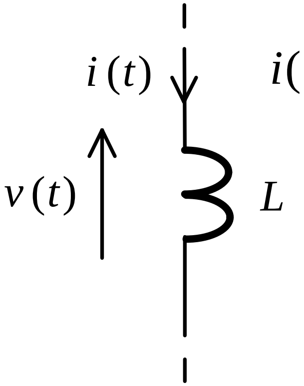
$$= \int_{-\infty}^{t} i(x)L\frac{di(x)}{dx}dx$$

$$= L\int_{-\infty}^{t} i(x)di(x) = \frac{L}{2}i^2(x)\Big|_{-\infty}^{t}$$

$$= \frac{L}{2}\left[i^2(t) - i^2(-\infty)\right]$$

$$= \frac{Li^2(t)}{2}, \qquad \text{if} \quad i(-\infty) = 0.$$

Now consider the operation of an inductor in an ac circuit:

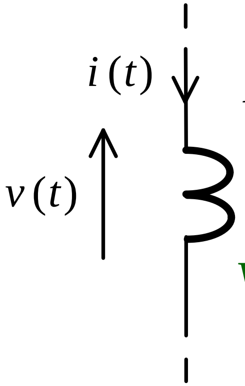


$$i(t) = r_i \sqrt{2} \cos(\omega t + \theta_i)$$

$$v(t) = L \frac{di(t)}{dt} = -\omega L r_i \sqrt{2} \sin(\omega t + \theta_i)$$

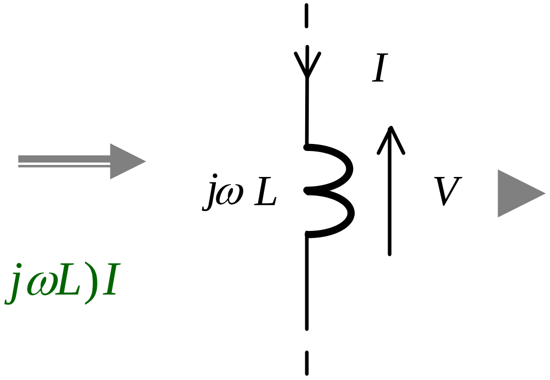
$$= \omega L r_i \sqrt{2} \cos(\omega t + \theta_i + \frac{\pi}{2})$$

In phasor:



$$I = r_i e^{j\theta_i}$$

$$V = \omega L r_i e^{j\theta_i} e^{j\pi/2} = j\omega L r_i e^{j\theta_i} = (j\omega L) I$$



$$\boxed{Z_L = \frac{V}{I} = j\omega L}$$

Z_L is the impedance of the inductor. The average power absorbed by the inductor:

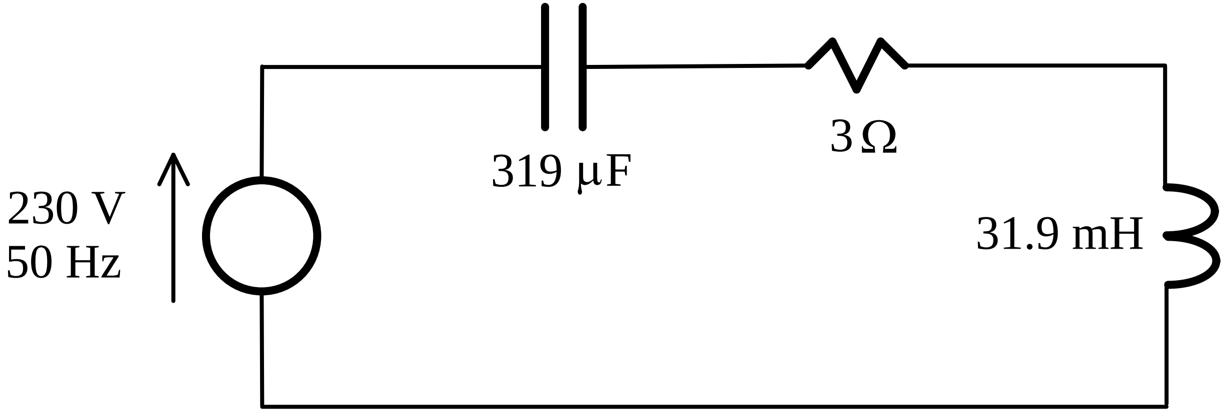
$$p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* Z_L I\right] = \mathrm{Re}\left[j\omega L I^* I\right] = \mathrm{Re}\left[j\omega L \left| I \right|^2\right] = 0$$

Since the phase of I relative to that of V is

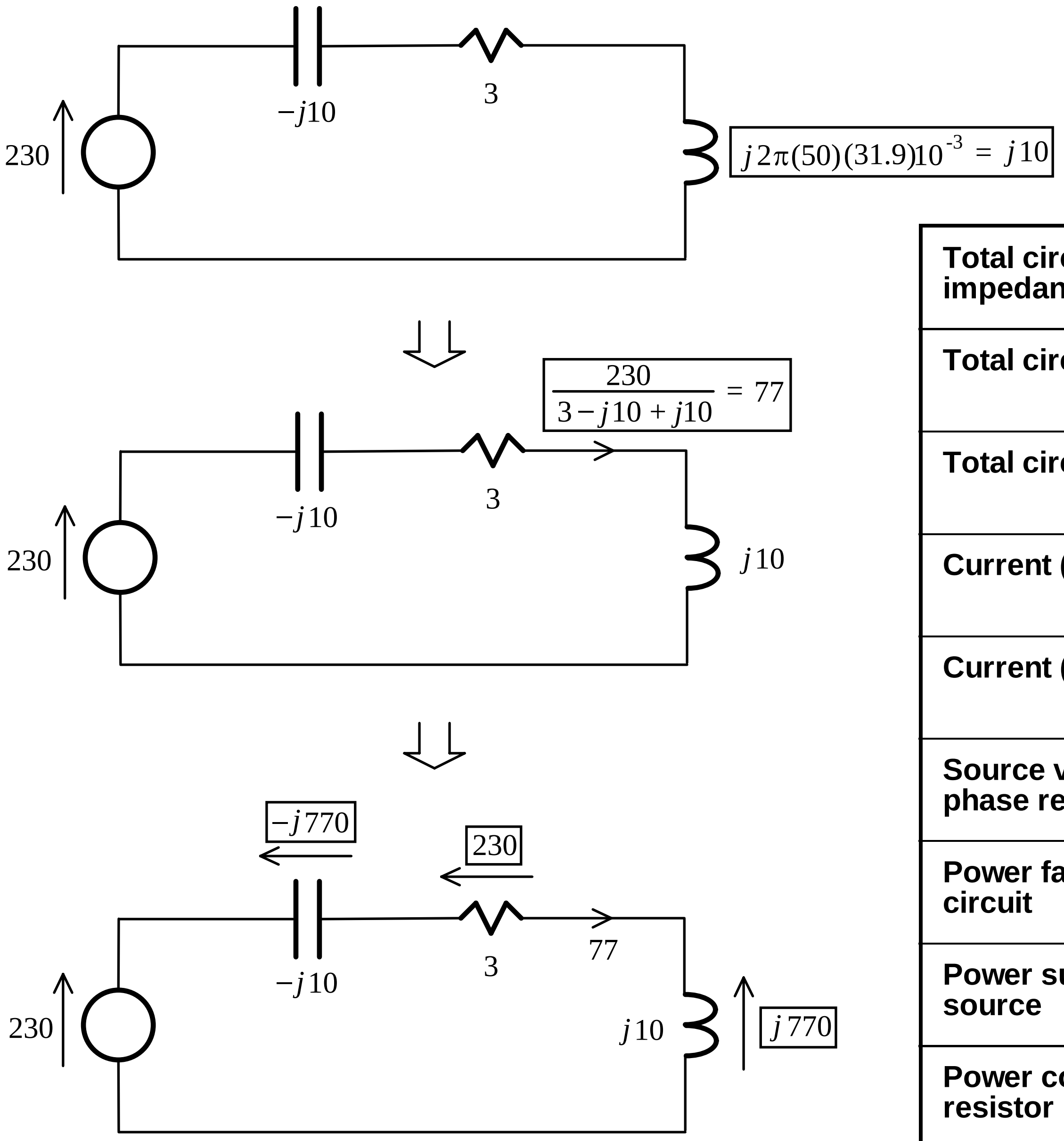
$$\mathrm{Arg}[I]-\mathrm{Arg}[V]=\mathrm{Arg}\left[\frac{I}{V}\right]=\mathrm{Arg}\left[\frac{1}{Z_L}\right]=\mathrm{Arg}\left[\frac{1}{j\omega L}\right]=-90^0$$

the ac current $i(t)$ lags the voltage $v(t)$ by 90°.

As an example, consider the following series ac circuit:



We can use the phasor representation to convert this ac circuit to a ‘DC’ circuit with complex voltage and resistance.



Summary of the circuit:

Total circuit impedance	$Z = 3 - j10 + j10 = 3\Omega$
Total circuit reactance	$X = \mathrm{Im}[Z] = \mathrm{Im}[3] = 0\Omega$
Total circuit resistance	$R = \mathrm{Re}[Z] = \mathrm{Re}[3] = 3\Omega$
Current (rms)	$\lvert I \rvert = 77\mathrm{A}$
Current (peak)	$\sqrt{2}\lvert I \rvert = 77\sqrt{2} = 108\mathrm{A}$
Source voltage-current phase relationship	$0\,(\text{in phase})$
Power factor of entire circuit	$\cos(0) = 1$
Power supplied by source	$\mathrm{Re}\left[(77)^{*}(230)\right] = 18\,\mathrm{kW}$
Power consumed by resistor	$\mathrm{Re}\left[(77)^{*}(3 \times 77)\right] = 18\,\mathrm{kW}$

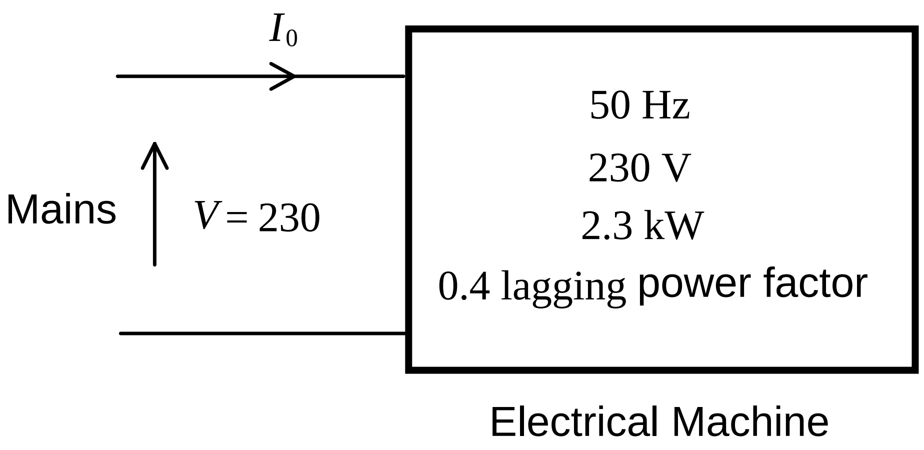
Note that the rms voltages across the inductor and capacitor are larger than the source voltage. This is possible in ac circuits because the reactances of capacitors and inductors, and so the voltages developed across them, may cancel out one another:

$$\boxed{\begin{array}{c}\text{Source}\\\text{voltage}\\230\end{array}} = \boxed{\begin{array}{c}\text{Voltage}\\\text{across}\\\text{capacitor}\\-j770\end{array}} + \boxed{\begin{array}{c}\text{Voltage}\\\text{across}\\\text{resistor}\\230\end{array}} + \boxed{\begin{array}{c}\text{Voltage}\\\text{across}\\\text{inductor}\\j770\end{array}}$$

In dc circuits, it is not possible for a passive resistor (with positive resistance) to cancel out the effect of another passive resistor (with positive resistance).

3.8 Power Factor Improvement

Consider the following system:



Due to the small power factor, the machine cannot be connected to standard 13A outlets even though it consumes only 2.3 kW of power.

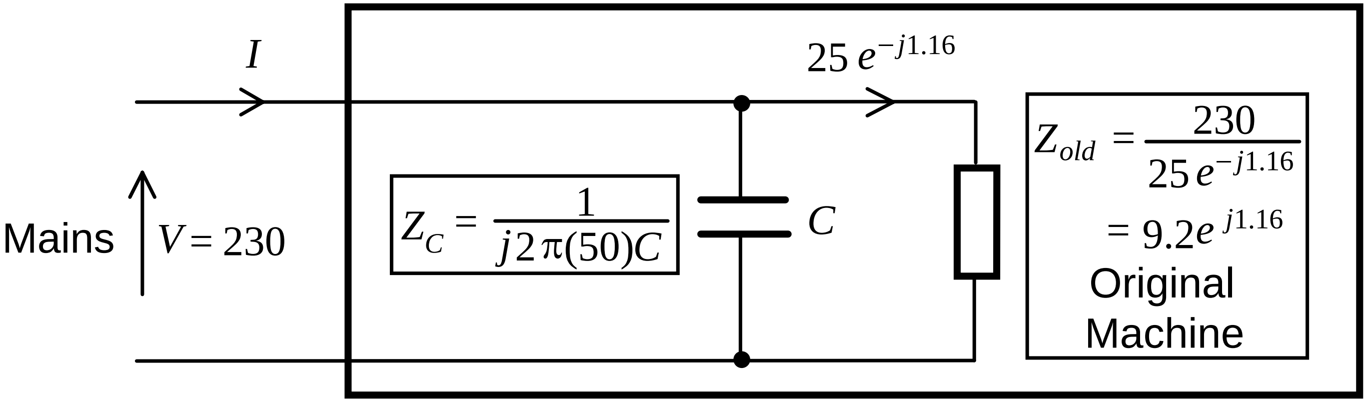
The current I_0 can be found as follows:

$$\frac{2300\text{W}}{(230\text{V})(|I_0|\text{A})} = 0.4 \Rightarrow |I_0| = \frac{2300}{(230)(0.4)} = 25$$

Can we improve it?

$$\left.\begin{array}{l} \cos\{\text{Arg}[I_0] - \text{Arg}[V]\} = 0.4 \\ \text{Arg}[I_0] - \text{Arg}[V] < 0 \end{array}\right\} \Rightarrow \text{Arg}[I_0] = -\cos^{-1}(0.4) = -1.16 \quad \Longrightarrow \quad \boxed{I_0 = |I_0| e^{j\text{Arg}[I_0]} = 25 e^{-j1.16}}$$

To overcome this problem, a parallel capacitor can be used to improve the power factor:



New Electrical Machine

$$I = \frac{V}{Z_C} + 25e^{-j1.16} = j23000\pi C + 10 - j23 = 10 + j(23000\pi C - 23)$$

Thus, if we choose $23000\pi C = 23 \Rightarrow C = 0.32\text{mF}$ then $I = 10\text{A}$ and

$$\text{Power factor of new machine} = \cos[\mathrm{Arg}(I) - \mathrm{Arg}(V)] = 1$$

By changing power factor, the improved machine can now be connected to standard 13A outlets. The price to pay is the use of an additional capacitor.

To reduce cost, we may wish to use a capacitor which is as small as possible. To find the smallest capacitor that will satisfy the 13A requirement:

$$|I|^2 = 10^2 + (72200C-23)^2 = 13^2 \quad \Longrightarrow \quad 13^2 = 10^2 + (72200C-23)^2$$

$$\Downarrow$$

$$0 = 10^2 - 13^2 + (72200C - 23)^2 = (72200C - 23)^2 - 8.3^2$$

$$\Downarrow$$

$$0 = (72200C - 23 - 8.3)(72200C - 23 + 8.3)$$

$$\Downarrow$$

$$C = 0.2\,\mathrm{mF} \quad \text{or} \quad 0.44\,\mathrm{mF}$$

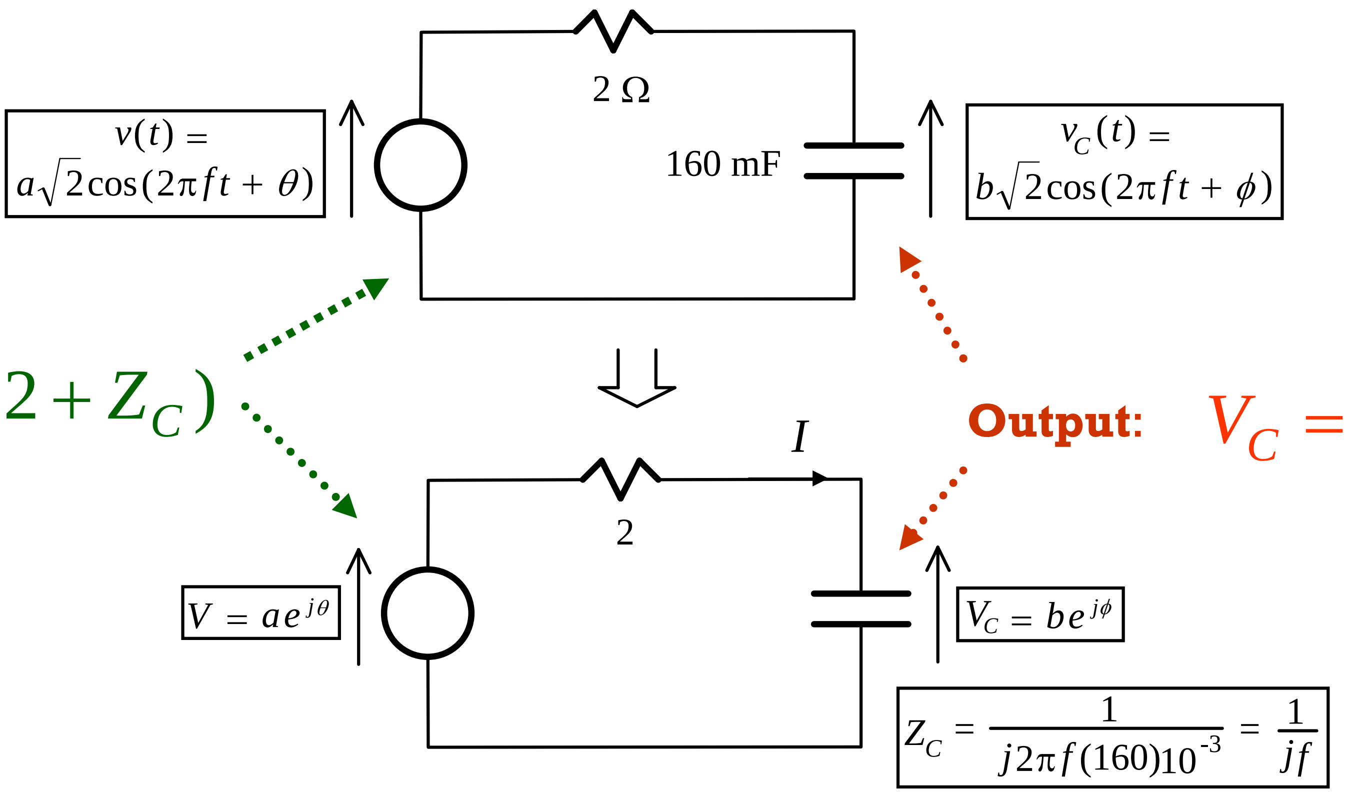
There are 2 possible values for *C*, one giving a lagging overall power factor, the other giving a leading overall power factor. To save cost, *C* should be

$$\boxed{C = 0.2\,\mathrm{mF}}$$

4. Frequency Response

4.1 RC Circuit

Consider the series RC circuit:



Input: $V = I(2 + Z_C)$

Output: $V_C = IZ_C$

$$H(f) = \frac{V_C}{V} = \frac{be^{j\phi}}{ae^{j\theta}} = \frac{b}{a}e^{j(\phi-\theta)} = \frac{Z_C}{2+Z_C} = \frac{\frac{1}{jf}}{2+\frac{1}{jf}} = \frac{1}{1+j2f}$$

⇐ **Frequency Response**

The magnitude of $H(f)$ is

$$|H(f)| = \left|\frac{V_C}{V}\right| = \frac{|V_C|}{|V|} = \frac{b}{a}$$

$$= \sqrt{\frac{1}{1+(2f)^2}} = \sqrt{\frac{1}{1+4f^2}}$$

and is called the ***magnitude response***.

The phase of $H(f)$ is

$$\mathrm{Arg}[H(f)] = \mathrm{Arg}\left[\frac{V_C}{V}\right] = \mathrm{Arg}[V_C] - \mathrm{Arg}[V]$$

$$= \phi - \theta = \mathrm{Arg}\left[\frac{1}{1+j2f}\right]$$

$$= -\mathrm{Arg}[1+j2f] = -\tan^{-1}(2f)$$

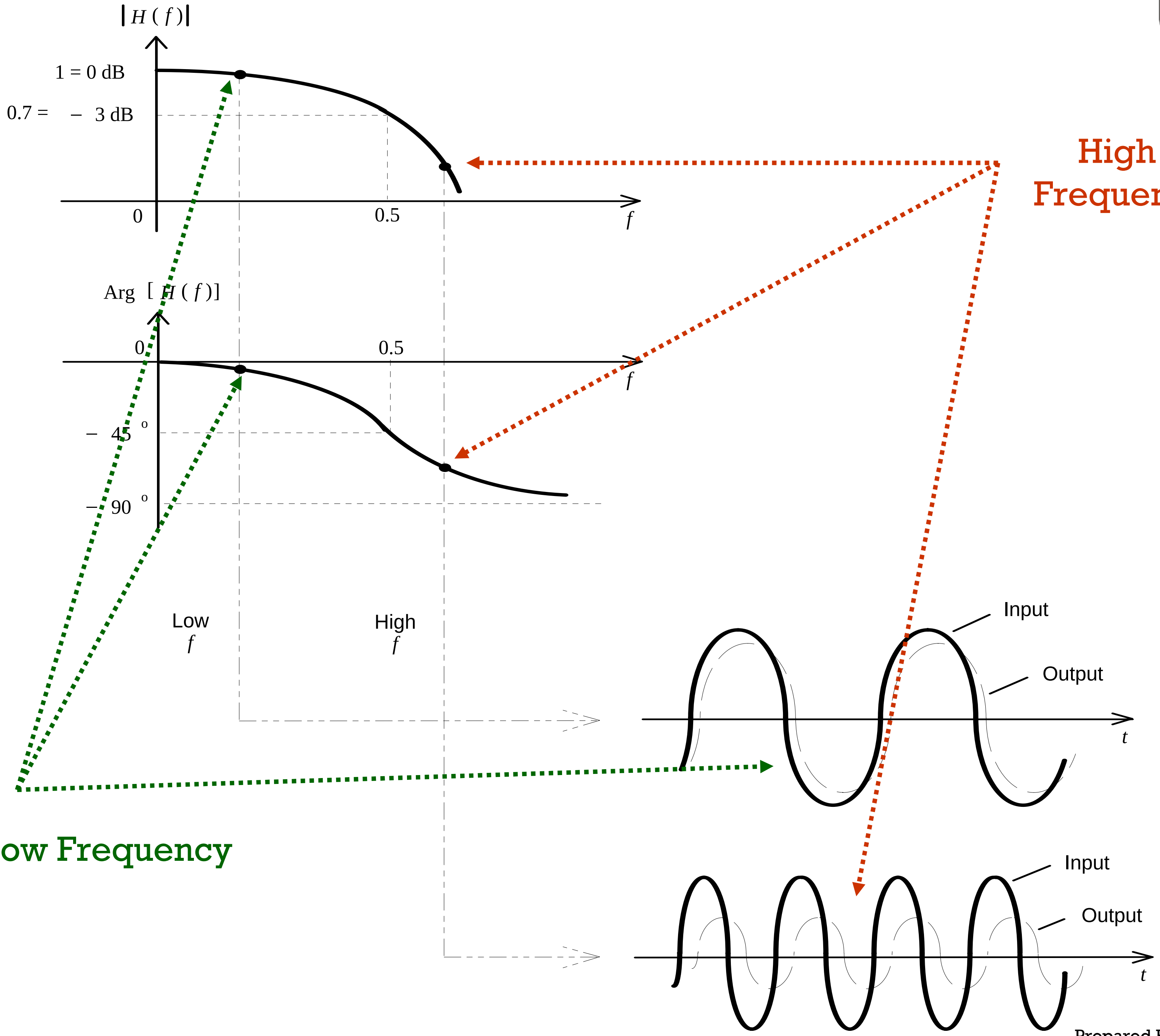
and is called the ***phase response***.

The physical significance of these responses is that $H(f)$ gives the ratio of output to input phasors, $|H(f)|$ gives the ratio of output to input magnitudes, and $\mathrm{Arg}[H(f)]$ gives the output to input phase difference at a frequency f.

Input	$v(t)=3\sqrt{2}\cos[2\pi(5)t+7]$ $V=3e^{j7}$	$v(t)=r\sin[2\pi(4)t]=r\cos\left[2\pi(4)t-\frac{\pi}{2}\right]$ $V=\frac{r}{\sqrt{2}}e^{-j\pi/2}$
Frequency	$f=5$	$f=4$
Frequency response	$H(5)=\frac{1}{1+j10}$	$H(4)=\frac{1}{1+j8}$
Magnitude response	$\lvert H(5)\rvert=\frac{1}{\sqrt{101}}$	$\lvert H(4)\rvert=\frac{1}{\sqrt{65}}$
Phase response	$\mathrm{Arg}[H(5)]=-\tan^{-1}(10)$	$\mathrm{Arg}[H(4)]=-\tan^{-1}(8)$
Output	$v_C(t)=\frac{3\sqrt{2}}{\sqrt{101}}\cos\left[2\pi(5)t+7-\tan^{-1}(10)\right]$ $V_C=\frac{3}{\sqrt{101}}e^{j\left[7-\tan^{-1}(10)\right]}$	$v_C(t)=\frac{r}{\sqrt{65}}\sin\left[2\pi(4)t-\tan^{-1}(8)\right]$ $=\frac{r}{\sqrt{65}}\cos\left[2\pi(4)t-\tan^{-1}(8)-\frac{\pi}{2}\right]$ $V_C=\frac{r}{\sqrt{130}}e^{j\left[-\tan^{-1}(8)-\pi/2\right]}$

Due to the presence of components such as capacitors and inductors with frequency-dependent impedances, $H(f)$ is usually frequency-dependent and the characteristics of the circuit is often studied by finding how $H(f)$ changes as f is varied. Numerically, for the series RC circuit:

f	$\lvert H(f)\rvert = \dfrac{1}{\sqrt{(1+4f^2)}}$	$\mathrm{Arg}[H(f)] = -\tan^{-1}(2f)$
0	$1 = 20\log(1) = 0\ \mathrm{dB}$	$0\ \mathrm{rad} = 0^0$
0.5	$\dfrac{1}{\sqrt{1+4(0.5)^2}} = \dfrac{1}{\sqrt{2}} = 20\log\left(\dfrac{1}{\sqrt{2}}\right) = -3\mathrm{dB}$	$-\tan^{-1}(2\times 0.5) = -\dfrac{\pi}{4}\mathrm{rad} = -45^0$
$\rightarrow\infty$	$\rightarrow 0 = -\infty\mathrm{dB}$	$-\tan^{-1}(\infty) = -\dfrac{\pi}{2}\mathrm{rad} = -90^0$



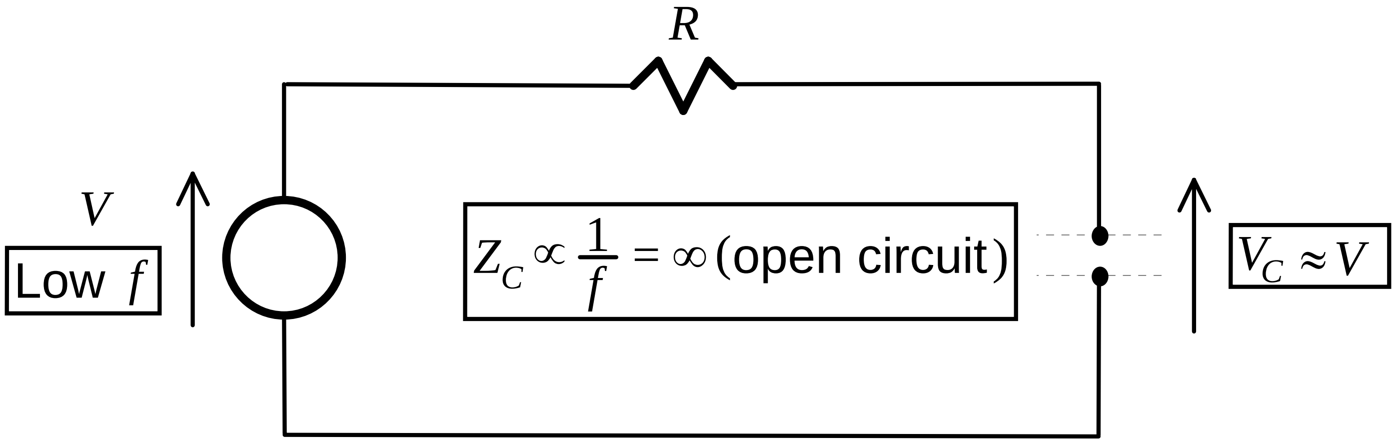
High
Frequency

Low Frequency

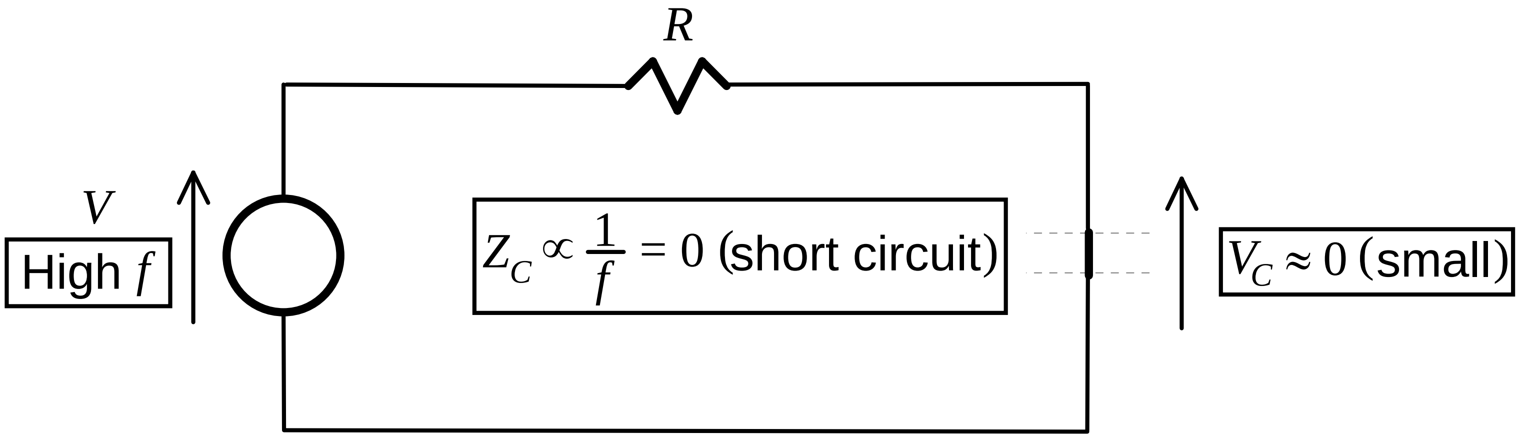
At small f, the output approximates the input. However, at high f, the output will become attenuated. Thus, the circuit has a ***low pass*** characteristic (low frequency input will be passed, high frequency input will be rejected).

The frequency at which $|H(f)|$ falls to –3 dB of its maximum value is called the ***cutoff*** frequency. For the above example, the cutoff frequency is 0.5 Hz.

To see why the circuit has a low pass characteristic, note that at low f, C has large impedance (approximates an open circuit) when compare with R (2 in the above example). Thus, V_C will be approximately equal to V :



However, at high f, C has small impedance (approximates a short circuit) when compare with R. Thus, V_C will be small:



Key Notes: **The capacitor is acting like a short circuit at high frequencies and an open circuit at low frequencies. It is totally open for a dc circuit.**

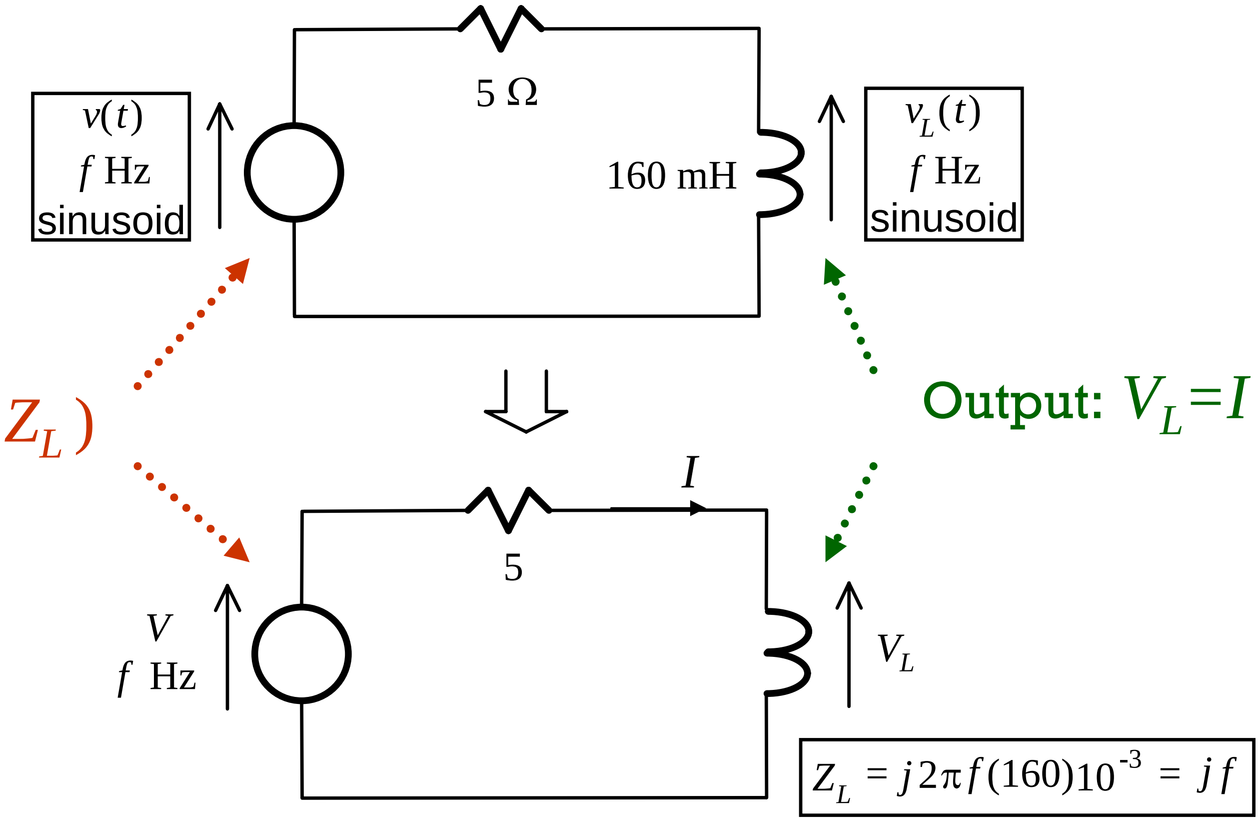
An Electric Joke

Q: Why does a capacitor block DC but allow AC to pass through?

A: You see, a capacitor is like this —| |— , OK. DC Comes straight, like this ——, and the capacitor stops it. But AC, goes up, down, up and down and jumps right over the capacitor!

4.2 RL Circuit

Consider the ***series RL circuit***:



Input: $V=I\,(\,5\,+\,Z_L\,)$

Output: $V_L=I\,Z_L$

$$Z_L = j2\pi f(160)10^{-3} = jf$$

$$\frac{V_L}{V} = H(f) = \frac{Z_L}{5+Z_L} = \frac{jf}{5+jf}$$

Frequency Response

The magnitude response is

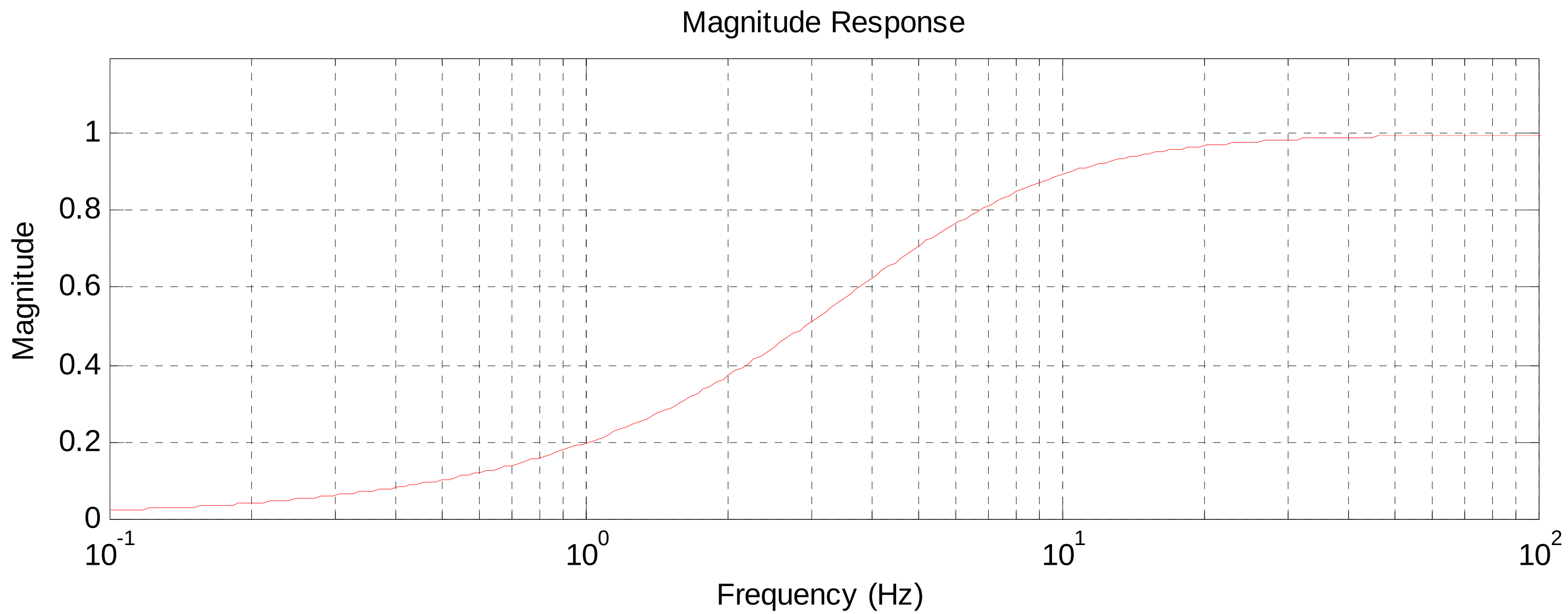
$$|H(f)| = \sqrt{\frac{f^2}{5^2+f^2}} = \sqrt{\frac{f^2}{25+f^2}}$$

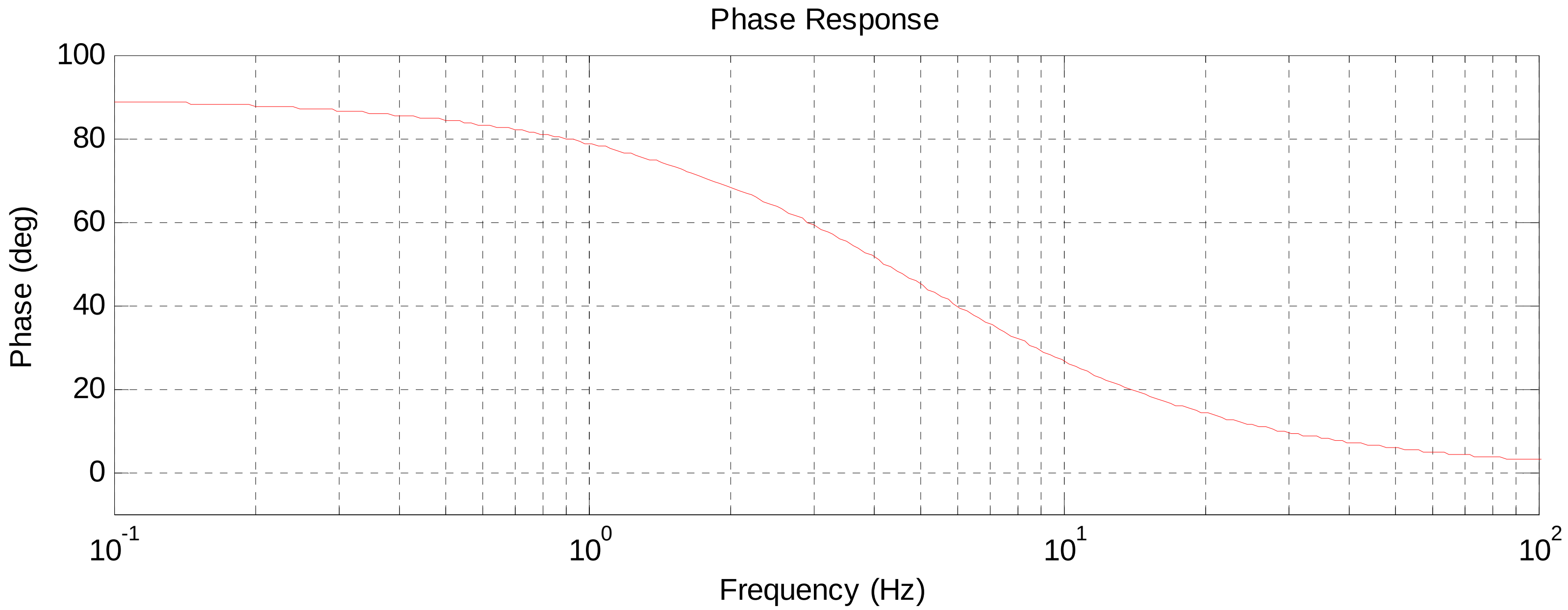
The phase response is

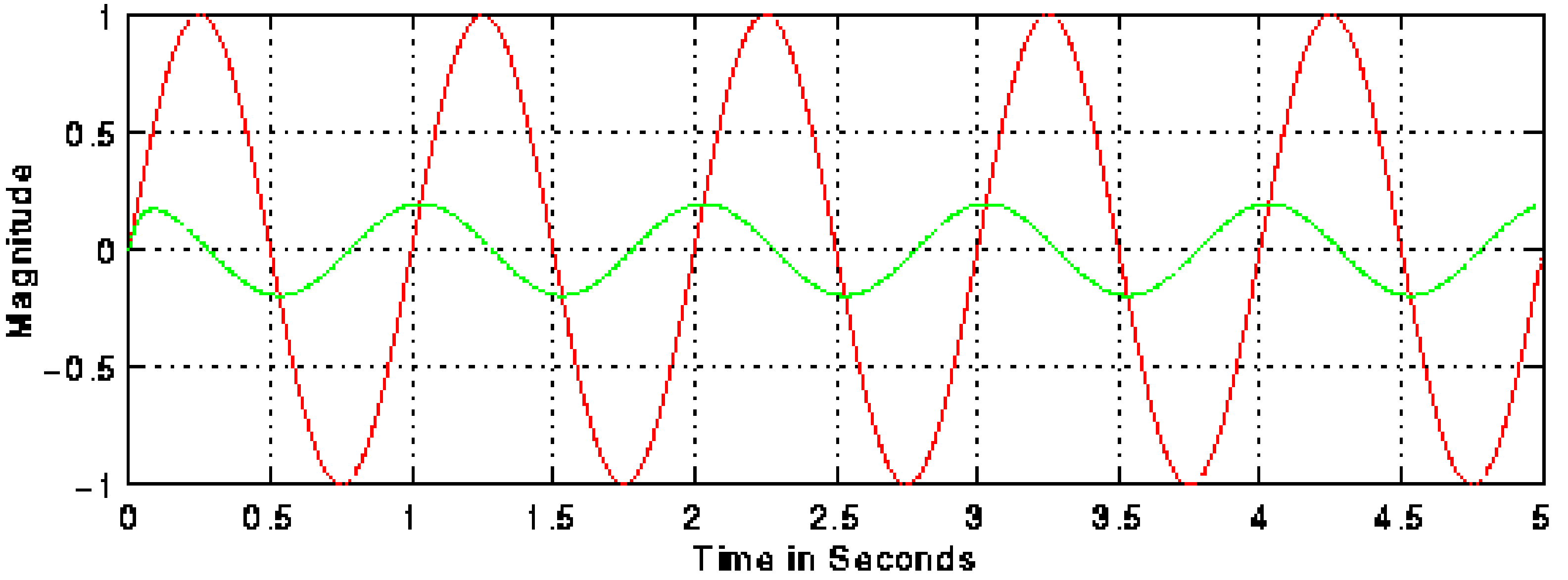
$$\operatorname{Arg}[H(f)] = \operatorname{Arg}\left[\frac{jf}{5+jf}\right]$$
$$= \operatorname{Arg}[jf] - \operatorname{Arg}[5+jf]$$
$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{5}\right)$$

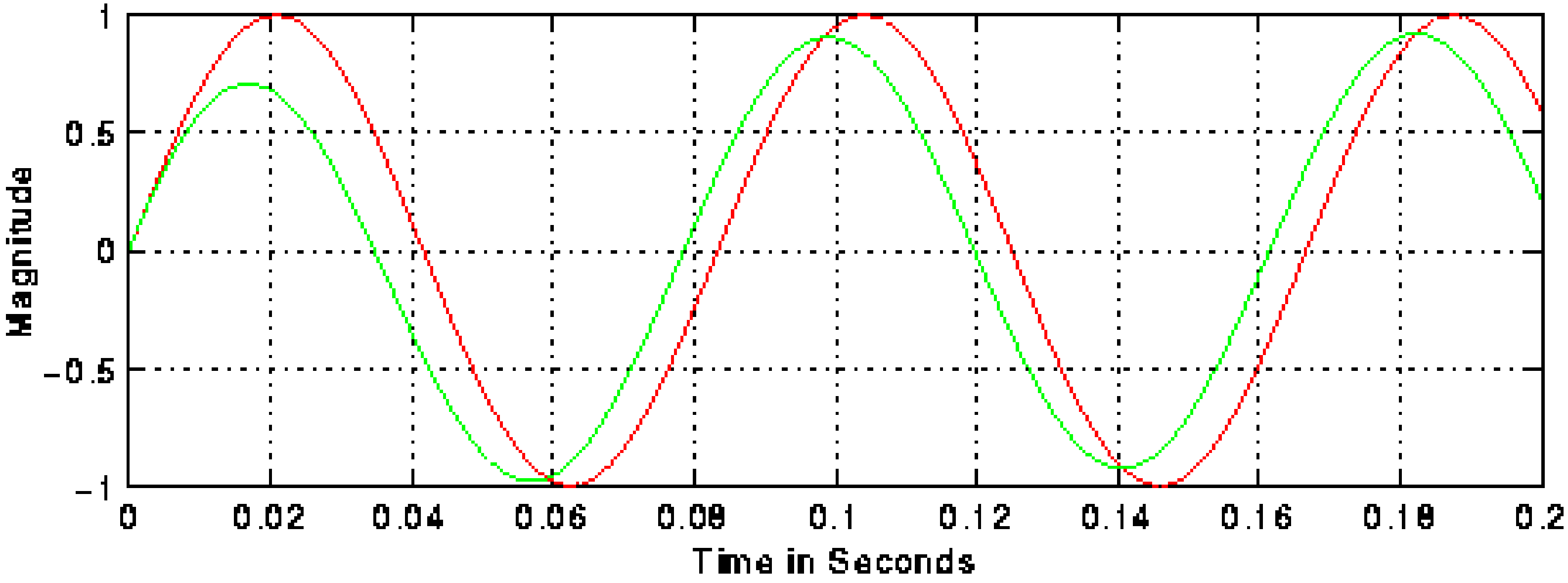
Numerically:

f	$\lvert H(f)\rvert = \sqrt{\frac{f^2}{25+f^2}}$	$\operatorname{Arg}[H(f)] = \frac{\pi}{2} - \tan^{-1}\left(\frac{f}{5}\right)$
0	$0 = 20\log(0) = -\infty \text{ dB}$	$\frac{\pi}{2} \text{ rad} = 90^0$
5	$\sqrt{\frac{5^2}{25+5^2}} = \frac{1}{\sqrt{2}} = 20\log\left(\frac{1}{\sqrt{2}}\right) = -3\text{dB}$	$\frac{\pi}{2} - \tan^{-1}\left(\frac{5}{5}\right) = \frac{\pi}{4} \text{ rad} = 45^0$
$\to \infty$	$\to 1 = 0 \text{ dB}$	$\frac{\pi}{2} - \tan^{-1}(\infty) = 0 \text{ rad} = 0^0$

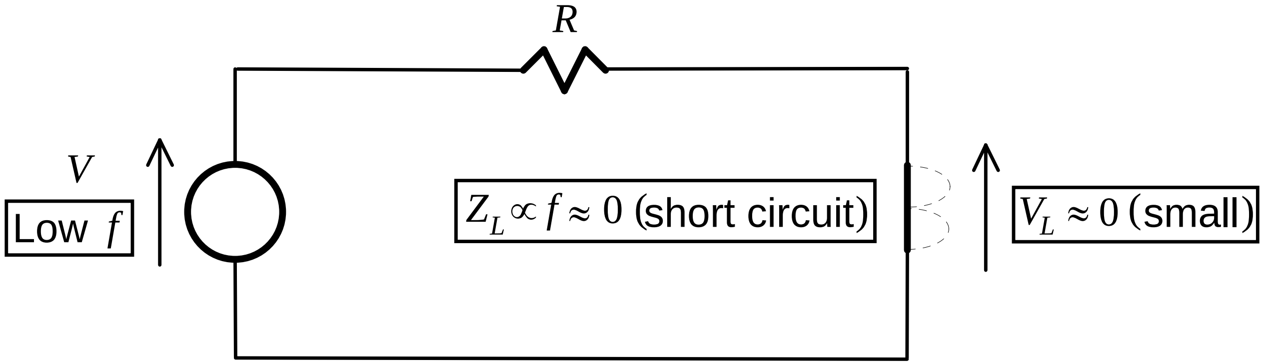

Magnitude Response
Magnitude
1
0.8
0.6
0.4
0.2
0
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (Hz)


Phase Response
Phase (deg)
100
80
60
40
20
0
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (Hz)

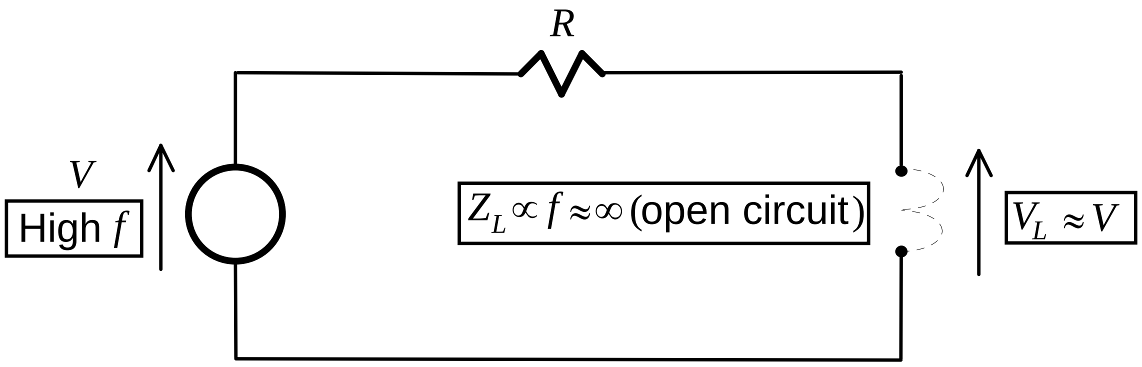

Red – Sine Input with f=1Hz; Green – Output


Red – Sine Input with f=12Hz; Green – Output

Physically, at small f, L has small impedance (approximates a short circuit) when compare with R (5 in the above example). Thus, V_L will be small:

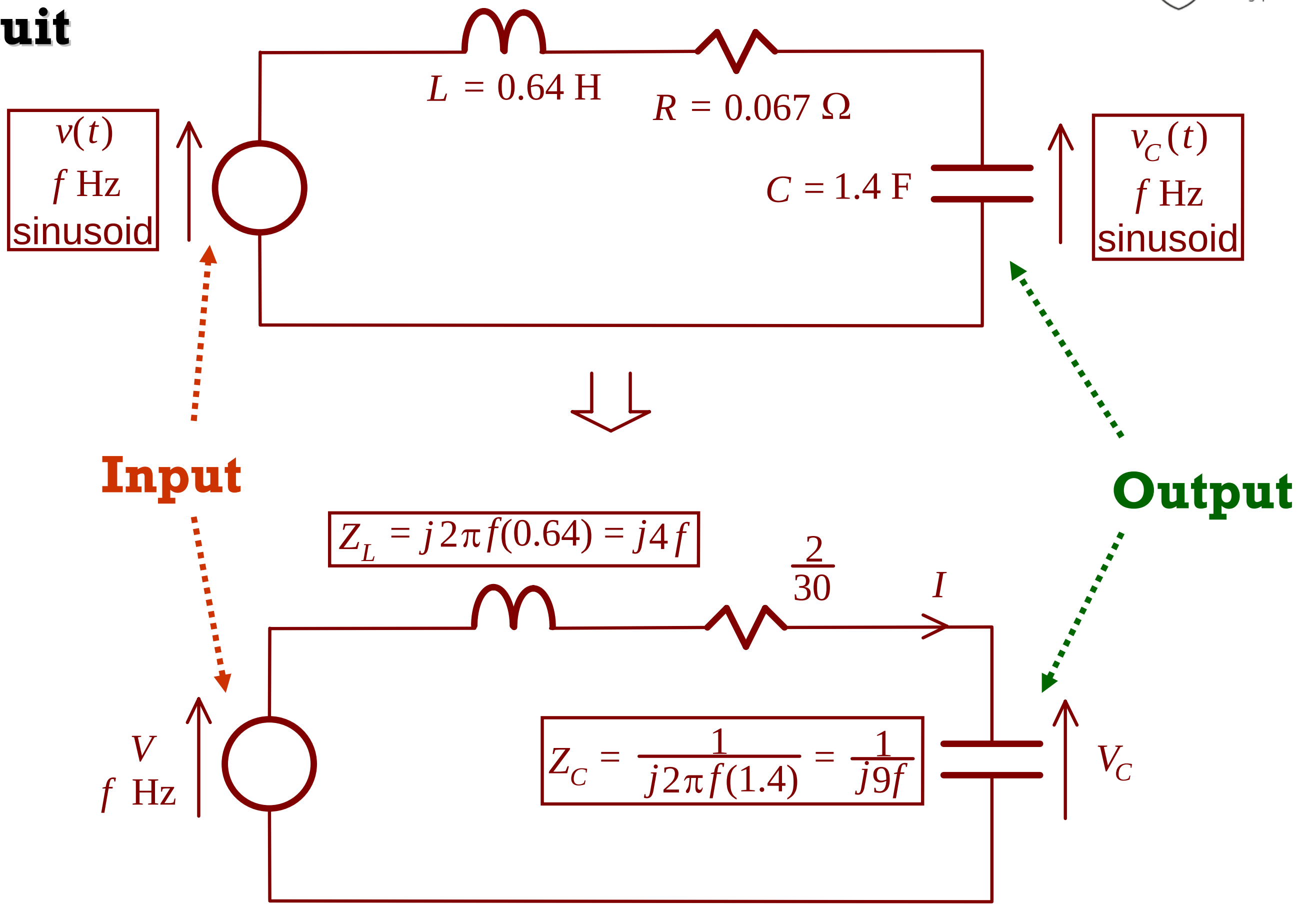


However, at high f, L has large impedance (approximates an open circuit) when compare with R. Thus, V_L will approximates V:



Due to these characteristics, the circuit is *highpass* in nature.

4.3 Series Tune Circuit



The total impedance is

$$Z = R+Z_L+Z_C = \frac{2}{30} + j4f + \frac{1}{j9f} = \frac{2}{30} + j\left(4f - \frac{1}{9f}\right) = \frac{2}{30}$$

Resonance Frequency ⟹

$$f_0 = \frac{1}{\sqrt{(4)(9)}} = \frac{1}{6} \Leftrightarrow f_0 = \frac{1}{\sqrt{(2\pi L)(2\pi C)}} = \frac{1}{2\pi\sqrt{LC}}$$

***Q* factor** ⟹

$$Q = \frac{\text{Reactance of inductor at } f_0}{\text{Resistance}} = \frac{2/3}{2/30} = 10 \Leftrightarrow Q = \frac{2\pi f_0 L}{R}$$

The frequency response is

$$H(f)=\frac{V_C}{V}=\frac{Z_C}{Z}=\frac{\dfrac{1}{j9f}}{\dfrac{2}{30}+j4f+\dfrac{1}{j9f}}=\frac{1}{1-36f^2+j0.6f}$$

The magnitude response is

$$|H(f)|=\frac{1}{\sqrt{\left(1-36f^2\right)^2+\left(0.6f\right)^2}}=\frac{1}{\sqrt{\left(36f^2\right)^2-\left(72-0.6^2\right)f^2+1}}$$

$$=\frac{1}{\sqrt{\left(36f^2\right)^2-2\left(36f^2\right)\left(1-\dfrac{0.6^2}{72}\right)+\left(1-\dfrac{0.6^2}{72}\right)^2+1-\left(1-\dfrac{0.6^2}{72}\right)^2}}$$

$$=\frac{1}{\sqrt{\left[36f^2-\left(1-\dfrac{0.6^2}{72}\right)\right]^2+\left(\dfrac{0.6^2}{72}\right)\left(2-\dfrac{0.6^2}{72}\right)}}$$

$$x^2-bx+c$$

$$=x^2-2\left(\frac{b}{2}\right)x+\left(\frac{b}{2}\right)^2$$

$$-\left(\frac{b}{2}\right)^2+c$$

$$=\left(x-\frac{b}{2}\right)^2+c-\left(\frac{b}{2}\right)^2$$

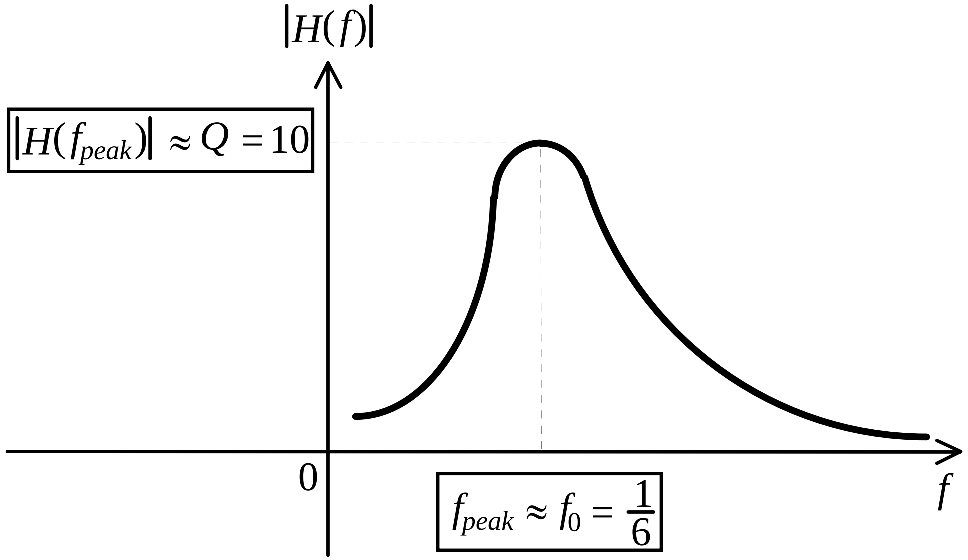
Since f only appears in the $[\bullet]^2$ term in the denominator and $[\bullet]^2 >= 0$, $|H(f)|$ will increase if $[\bullet]^2$ becomes smaller, and vice versa.

The maximum value for $|H(f)|$ corresponds to the situation of [•]^2 or at a frequency $f = f_{\text{peak}}$ given by:

$$36 f_{peak}^2 = 1 - \frac{0.6^2}{72} \approx 1 \quad \Leftrightarrow \quad f_{peak} \approx \frac{1}{6} \quad \Leftrightarrow \quad f_{peak} \approx f_0$$

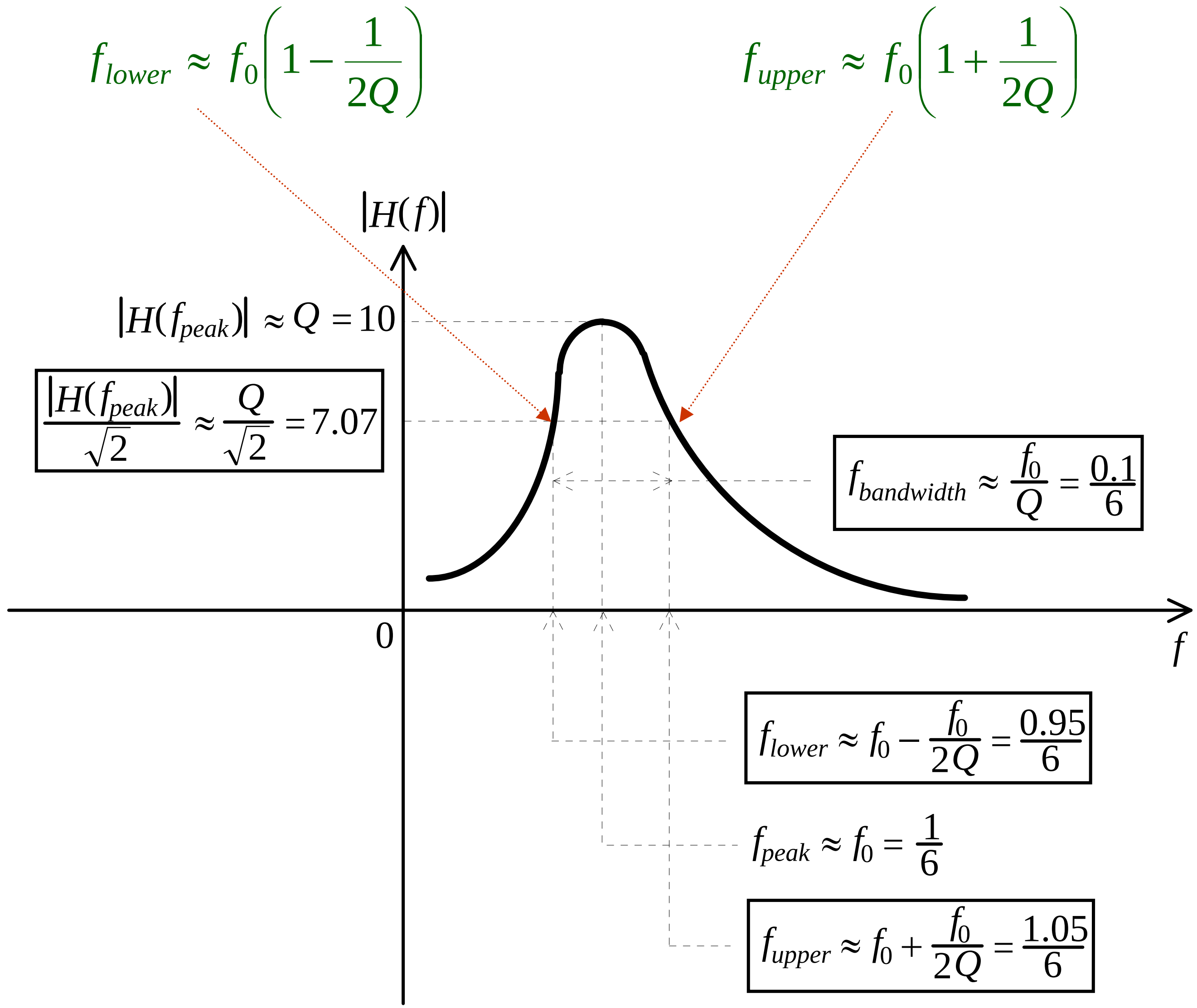
At $f = f_{\text{peak}}$, [•]^2 and the maximum value for $|H(f)|$ is

$$\left|H\left(f_{peak}\right)\right| = \frac{1}{\sqrt{\left(\frac{0.6^2}{72}\right)\left(2 - \frac{0.6^2}{72}\right)}} \approx \frac{1}{\sqrt{\left(\frac{0.6^2}{72}\right)(2)}} = 10 \quad \Leftrightarrow \quad \left|H\left(f_{peak}\right)\right| \approx Q$$



The series tuned circuit has a ***bandpass*** characteristic. Low- and high-frequency inputs will get attenuated, while inputs close to the resonant frequency will get amplified by a factor of approximately Q.

The cutoff frequencies, at which $|H(f)|$ decrease by a factor of 0.7071 or by 3 dB from its peak value $|H(f_{\text{peak}})|$, can be shown to be given by



$$f_{lower} \approx f_0\left(1-\frac{1}{2Q}\right)$$

$$f_{upper} \approx f_0\left(1+\frac{1}{2Q}\right)$$

$$|H(f_{peak})| \approx Q = 10$$

$$\boxed{\frac{|H(f_{peak})|}{\sqrt{2}} \approx \frac{Q}{\sqrt{2}} = 7.07}$$

$$\boxed{f_{bandwidth} \approx \frac{f_0}{Q} = \frac{0.1}{6}}$$

$$\boxed{f_{lower} \approx f_0 - \frac{f_0}{2Q} = \frac{0.95}{6}}$$

$$f_{peak} \approx f_0 = \frac{1}{6}$$

$$\boxed{f_{upper} \approx f_0 + \frac{f_0}{2Q} = \frac{1.05}{6}}$$

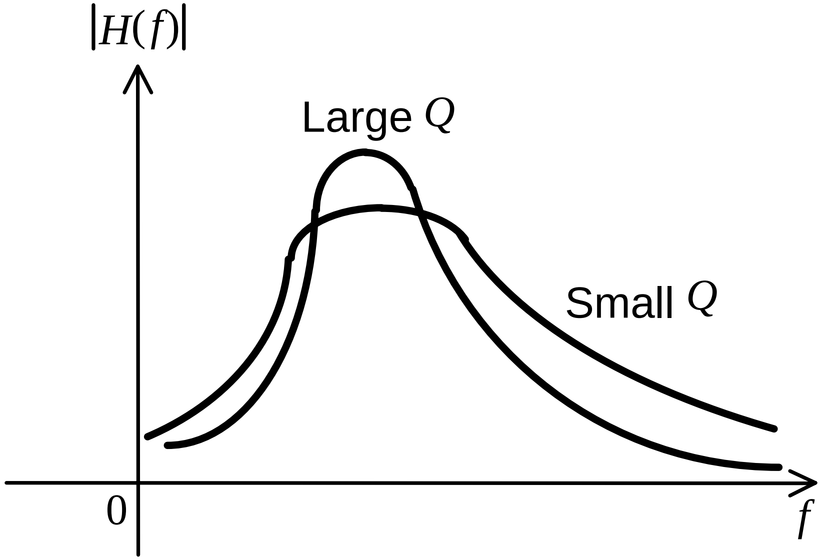
Very roughly, the circuit will pass inputs with frequency between f_{lower} and f_{upper}. The ***bandwidth*** of the circuit is

$$f_{bandwidth} = f_{upper} - f_{lower} \approx \frac{f_0}{Q}$$

and the ***fractional bandwidth*** is

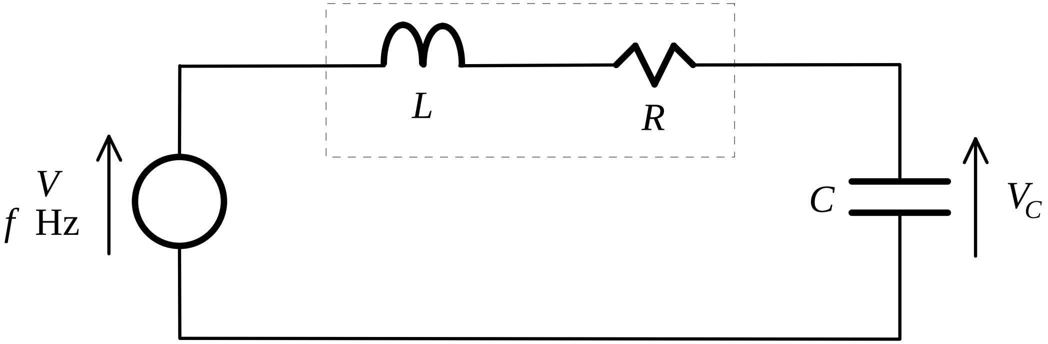
$$\frac{f_{bandwidth}}{f_0} \approx \frac{1}{Q}$$

The larger the Q factor, the sharper the magnitude response, the bigger the amplification, and the narrower the fractional bandwidth:



In practice, a series tune circuit usually consists of a practical inductor or coil connected in series with a practical capacitor. Since a practical capacitor usually behaves quite closely to an ideal one but a coil will have winding resistance, such a circuit can be represented by:

Equivalent circuit for coil or practical inductor



The main features are:

Circuit impedance	$Z = R + j2\pi f L + \dfrac{1}{j2\pi f C}$
Resonance frequency	$f_0 = \dfrac{1}{2\pi\sqrt{LC}}$
Q **factor**	$Q = \dfrac{2\pi f_0 L}{R}$
Frequency response	$H(f) = \dfrac{1}{1 - 4\pi^2 f^2 LC + j2\pi f CR} = \dfrac{1}{1 - \left(\dfrac{f}{f_0}\right)^2 + \dfrac{j}{Q}\left(\dfrac{f}{f_0}\right)}$

For the usual situation when *Q* is large:

Magnitude response	Bandpass with $\lvert H(f)\rvert$ decreasing as $f \to 0$ and $f \to \infty$
Response peak	$\lvert H(f)\rvert$ peaks at $f = f_{peak} \approx f_0$ with $\left\lvert H\left(f_{peak}\right)\right\rvert \approx Q$
Cutoff frequencies	$\lvert H(f)\rvert = \dfrac{\left\lvert H\left(f_{peak}\right)\right\rvert}{\sqrt{2}} \approx \dfrac{Q}{\sqrt{2}}$ at $f = f_{lower}, f_{upper} \approx f_0\left(1-\dfrac{1}{2Q}\right), \; f_0\left(1+\dfrac{1}{2Q}\right)$
Bandwidth	$f_{bandwidth} = f_{upper} - f_{lower} \approx \dfrac{f_0}{Q}$
Fractional bandwidth	$\dfrac{f_{bandwidth}}{f_0} \approx \dfrac{1}{Q}$

The Q factor is an important parameter of the circuit.

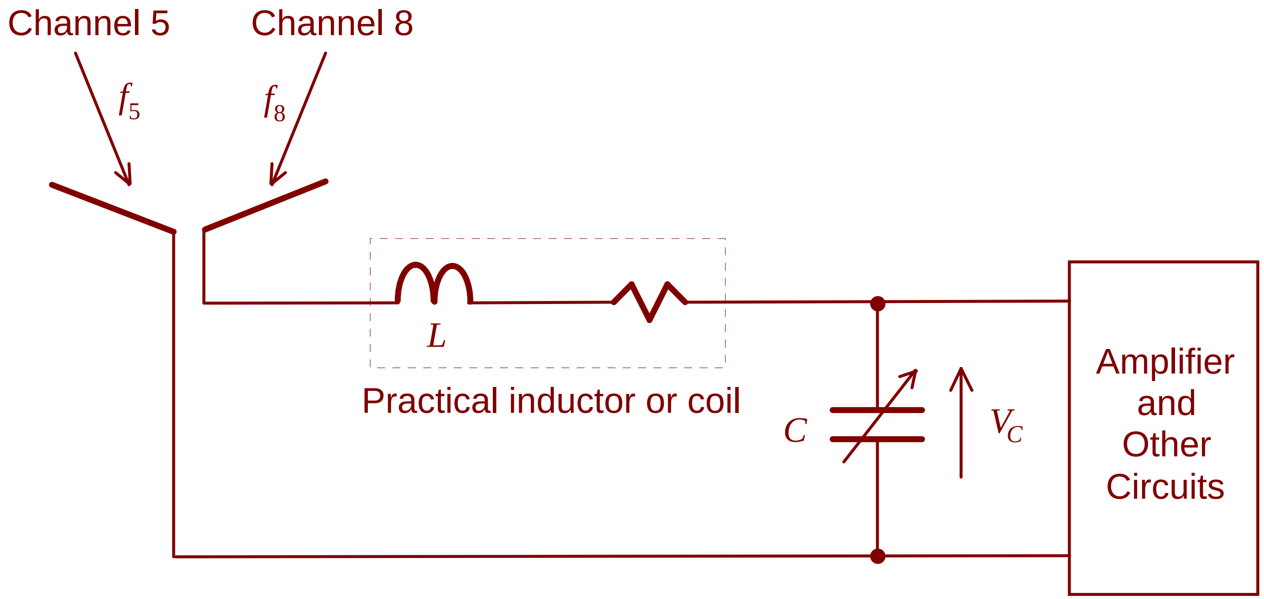
$$Q = \frac{2\pi f_0 L}{R} = \frac{\text{Inductor reactance at } f_0}{\text{Circuit resistance}}$$

However, since R is usually the winding resistance of the practical coil making up the tune circuit:

$$Q = \frac{\text{Reactance of practical coil at } f_0}{\text{Resistance of practical coil}}$$

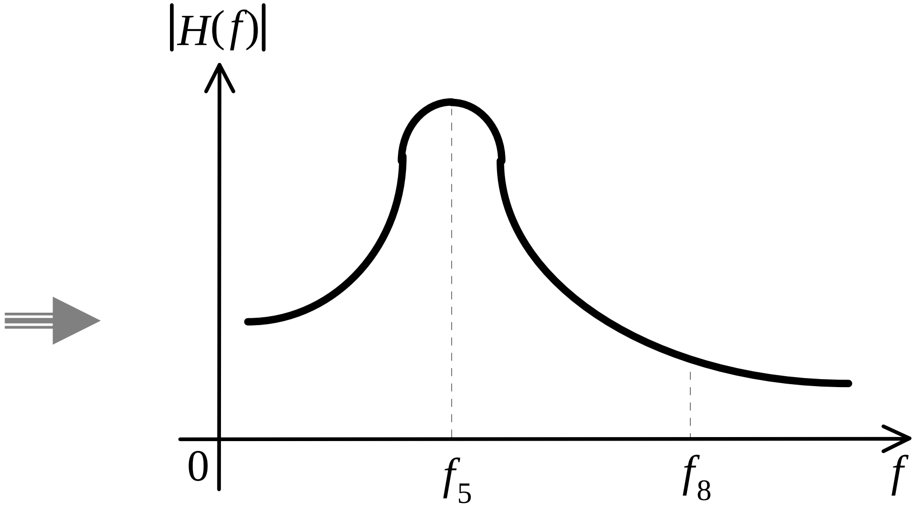
As a good practical coil should have low winding resistance and high inductance, the Q factor is often taken to be a characteristic of the practical inductor or coil. The higher the Q factor, the higher the ***quality*** of the coil.

Due to its bandpass characteristic, tune circuits are used in radio and TV tuners for selecting the frequency channel of interest:



To tune in to channel 5, C has to be adjusted to a value of C_5 so that the circuit resonates at a frequency given by

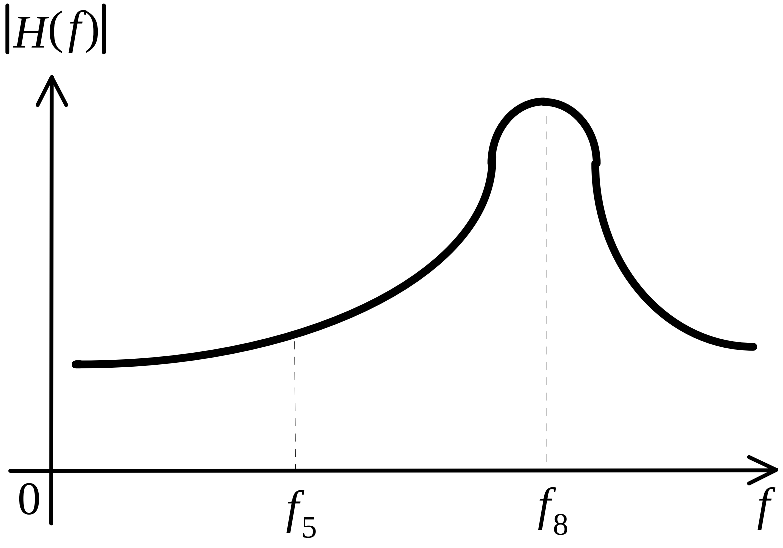
$$f_5 = \frac{1}{2\pi\sqrt{LC_5}}$$



To tune in to channel 8, C has to be adjusted to a value of C_8 so that the circuit resonates at a frequency given by

$$f_8 = \frac{1}{2\pi\sqrt{LC_8}}$$

and has a magnitude response of:



Additional Notes on Frequency Response

Frequency response is defined as the ratio of the phasor of the output to the phasor of the input. Note that both the input and output could be voltage and/or current. Thus, frequency response could have

$$\frac{V(\text{output})}{I(\text{input})}, \qquad \frac{V(\text{output})}{V(\text{input})}, \qquad \frac{I(\text{output})}{I(\text{input})}, \qquad \frac{I(\text{output})}{V(\text{input})}.$$

That's all, folks!

The End