E.E. Foundation

1

SI UNITS

1.1 Important Quantities and Base SI Units

Length	metre	m
Mass, *m*	kilogram	kg
Time, *t*	second	s
Electric current, *i*	ampere	A
Thermodynamic temperature	kelvin	K
Plane angle	radian	rad

1.2 Important Derived Quantities and SI Units

Force, f	$f = ma$	newton	$\mathrm{N} = \mathrm{kg\ m/s^2}$
Energy, e	$e = fd = mgh = \frac{mu^2}{2}$	joule	$\mathrm{J} = \mathrm{N\ m}$
Power, p	$p = \frac{de}{dt}$, $e = \int p dt$	watt	$\mathrm{W} = \mathrm{J/s}$
Electric charge, q	$q = \int i dt$, $i = \frac{dq}{dt}$	coulomb	$\mathrm{C} = \mathrm{A\ s}$
Electric potential, v	$v = \frac{de}{dq} = \frac{de}{dt}\frac{dt}{dq} = \frac{p}{i}$ $p = vi$, $e = \int v i dt$	volt	$\mathrm{V} = \mathrm{J/C} = \mathrm{W/A}$
Resistance, R	$v = iR$, $p = i^2 R = \frac{v^2}{R}$	ohm	$\Omega = \mathrm{V/A}$
Inductance, L	$v = L\frac{di}{dt}$, $e = \frac{Li^2}{2}$	henry	$\mathrm{H} = \mathrm{V\,s/A}$
Capacitance, C	$i = C\frac{dv}{dt}$, $q = Cv$, $e = \frac{Cv^2}{2}$	farad	$\mathrm{F} = \mathrm{A\,s/V}$
Magnetic flux, Φ	$v = N\frac{d\Phi}{dt}$	weber	$\mathrm{Wb} = \mathrm{V\,s}$
Magnetic flux density, B	$B = \frac{\Phi}{A}$	tesla	$\mathrm{T} = \mathrm{Wb/m^2}$

1.3 Important Decimal Multiples and Sub-Multiples

10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f

1.4 References

[1] ISO Standard 31 (13 parts), International Organisation for Standardisation (ISO).

[2] Symbols and Abbreviations for use in Electrical and Electronic Engineering Courses, Institution of Electrical Engineers, London.

2

DC Circuit Analysis

2.1 Voltage Source

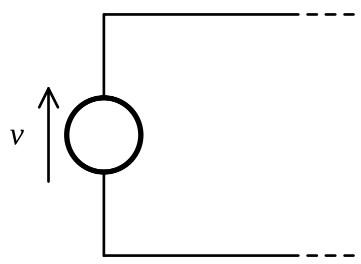
Two common dc (direct current) voltage sources are:

Dry battery (AA, D, C, etc.)	1.5 V
Lead acid battery in car	12 V

Regardless of the load connected and the current drawn, the above sources have the characteristic that the supply voltage will not change very much.

The definition for an ***ideal voltage source*** is thus one whose output voltage does not depend on what has been connected to it.

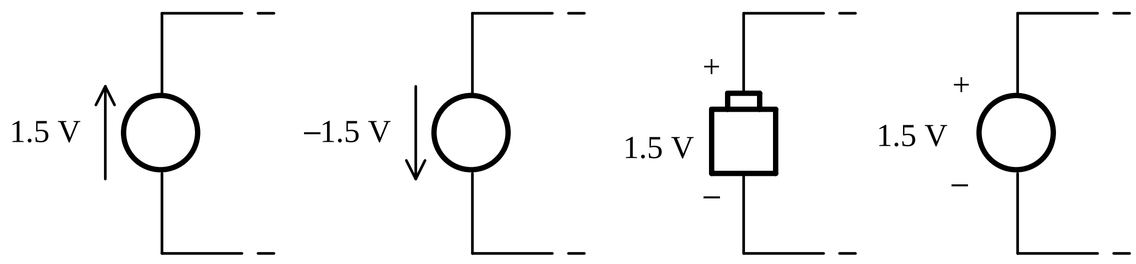
The circuit symbol is



Basically, the arrow and the value v signifies that the top terminal has a potential of v with respect to the bottom terminal regardless of what has been connected and the current being drawn.

Note that the current being drawn is not defined but depends on the load connected. For example, a battery (which approximates an ideal voltage source) will give no current if nothing is connected to it, but may be supplying a lot of current if a powerful motor is connected across its terminals. However, in both cases, the terminal voltages will be roughly the same.

Using the above and other common circuit symbol, the following sources are identical:

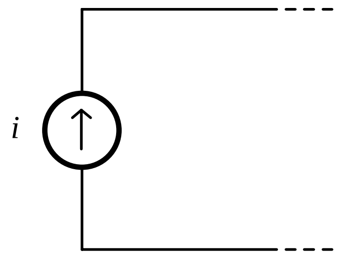


Note that on its own, the arrow does not correspond to the positive terminal. Instead, the positive terminal depends on both the arrow and the sign of the voltage v which may be negative.

2.2 Current Source

In the same way that the output voltage of an ideal voltage source does not depend on the load and the current drawn, the current delivers by an ***ideal current source*** does not depend on what has been connected and the voltage across its terminals.

Its circuit symbol is

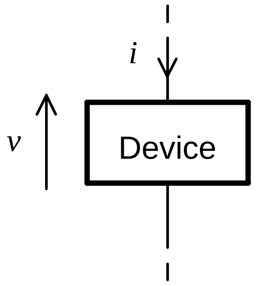


Note that ideal voltage and current sources are idealisations and do not exist in practice. Many practical electrical sources, however, behave like ideal voltage/current sources.

One practical source that approximates an ideal current source is the optical detector in an infra red remote control unit. The amount of current that the detector supply depends mainly on the intensity of light (number of photons) falling on the detector, but is quite independent on the voltage across the device.

2.3 Power and Energy

Consider the following device with a current of i :

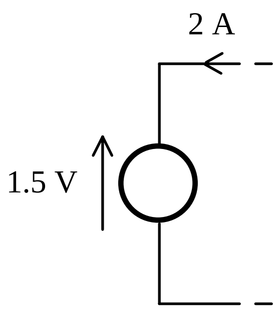


Power Consumed by Device $p = vi$

In 1s, there are i charges passing through the device. Their electric potential will decrease by v and their electric potential energy will decrease by iv. This energy will have been absorbed or consumed by the device.

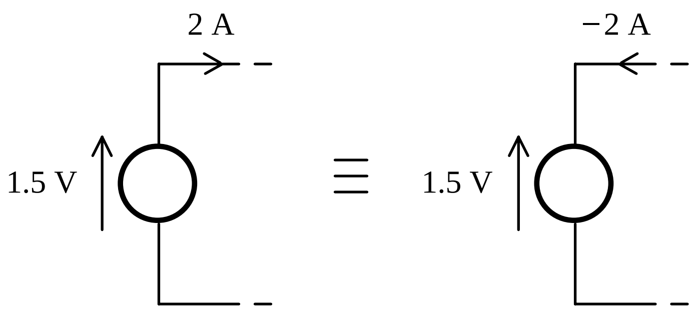
The ***power*** or the rate of energy ***consumed*** by the device is thus iv.

Note that $p = vi$ gives the power consumed by the device if the voltage and current arrows are opposite to one another. The following examples illustrate this point:



Power consumed/ absorbed by source $= 3\text{ W}$

Energy absorbed in 100 hr $= 300\text{ W hr}$ $= 0.3\text{ kW hr}$ $= 0.3\text{ unit}$ in PUB bill

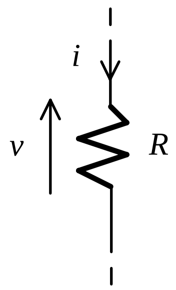


Power supplied by source $= 3\text{ W}$

Power absorbed by source $= -3\text{ W}$

2.4 Resistor

The symbol for an ideal resistor is



Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship follows Ohm's law:

$$v = iR$$

The power consumed is

$$p = vi = i^2 R = \frac{v^2}{R}$$

Common practical resistors are made of carbon film, wires, etc.

2.5 Relative Power

Powers, voltages and currents are often measured in relative terms with respect to certain convenient reference values. Thus, taking

$$p_{ref} = 1\,\text{mW}$$

as the reference, the power

$$p = 2\,\text{W}$$

will have a relative value of

$$\frac{p}{p_{ref}} = \frac{2\,\text{W}}{1\text{mW}} = \frac{2\,\text{W}}{10^{-3}\,\text{W}} = 2000$$

The $\log$ of this relative power or power ratio is usually taken and given a dimensionless unit of ***bel***. The power $p = 2\,\text{W}$ is equivalent to

$$\log\left(\frac{p}{p_{ref}}\right) = \log(2000) = \log(1000) + \log(2) = 3.3\text{bel}$$

As bel is a large unit, the finer sub-unit, ***decibel*** or ***dB*** (one-tenth of a Bel), is more commonly used. In dB, $p = 2\,\text{W}$ is the same as

$$10\log\left(\frac{p}{p_{ref}}\right) = 10\log(2000) = 33\text{dB}$$

As an example:

Reference	Actual power	Relative power	
p_{ref}	p	p/p_{ref}	$10\log(p/p_{ref})$
$1\,\text{mW}$	$1\,\text{mW}$	1	$0\,\text{dB}$
$1\,\text{mW}$	$2\,\text{mW}$	2	$3\,\text{dB}$
$1\,\text{mW}$	$10\,\text{mW}$	10	$10\,\text{dB}$
$1\,\text{mW}$	$20\,\text{mW}$	$20 = 10 \times 2$	$13\,\text{dB} = 10\,\text{dB} + 3\,\text{dB}$
$1\,\text{mW}$	$100\,\text{mW}$	100	$20\,\text{dB}$
$1\,\text{mW}$	$200\,\text{mW}$	$200 = 100 \times 2$	$23\,\text{dB} = 20\,\text{dB} + 3\,\text{dB}$

Although dB measures relative power, it can also be used to measure relative voltage or current which are indirectly related to power.

For instance, taking

$$v_{ref} = 0.1\,\text{V}$$

as the reference voltage, the power consumed by applying v_{ref} to a resistor R will be

$$p_{ref} = \frac{v_{ref}^{\,2}}{R}$$

Similarly, the voltage

$$v = 1\,\text{V}$$

will lead to a power consumption of

$$p = \frac{v^2}{R}$$

The voltage v relative to v_{ref} will then give rise to a relative power of

$$\frac{p}{p_{ref}} = \frac{\frac{v^2}{R}}{\frac{v_{ref}^{\,2}}{R}} = \left(\frac{v}{v_{ref}}\right)^2 = \left(\frac{1}{0.1}\right)^2 = 100$$

or in dB:

$$10\log\left(\frac{p}{p_{ref}}\right)\text{dB}=10\log\left(\frac{v}{v_{ref}}\right)^2\text{dB}=20\log\left(\frac{v}{v_{ref}}\right)\text{dB}=20\log\left(\frac{1}{0.1}\right)\text{dB}=20\text{dB}$$

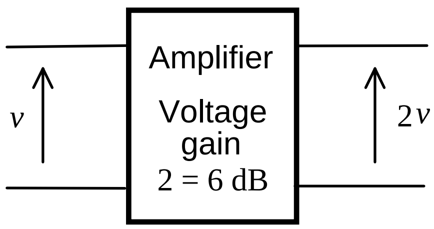
This is often used as a measure of the relative voltage v/v_{ref} .

As an example:

Reference	Actual voltage	Relative voltage	
v_{ref}	v	v/v_{ref}	$20\log(v/v_{ref})$
0.1V	0.1V	1	0dB
0.1V	$0.1\sqrt{2}\text{ V}$	$\sqrt{2}$	3dB
0.1V	0.2 V	$2=\sqrt{2}\times\sqrt{2}$	$6\text{dB}=3\text{dB}+3\text{dB}$
0.1V	$0.1\sqrt{10}\text{ V}$	$\sqrt{10}$	10dB
0.1V	$0.1\sqrt{20}\text{ V}$	$\sqrt{20}=\sqrt{10}\times\sqrt{2}$	$13\text{dB}=10\text{dB}+3\text{dB}$
0.1V	1V	$10=\sqrt{10}\times\sqrt{10}$	$20\text{dB}=10\text{dB}+10\text{dB}$

The measure of relative current is the same as that of relative voltage and can be done in dB as well.

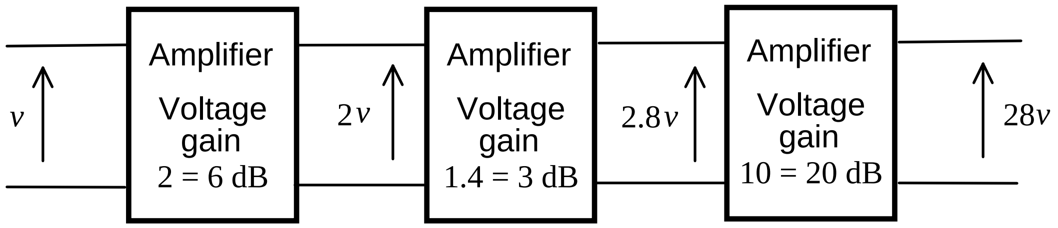
The advantage of measuring relative power, voltage and current in dB can be seen from considering the following voltage amplifier:



The voltage gain of the amplifier is given in terms of the output voltage relative to the input voltage or, more conveniently, in dB:

$$g=\frac{2v}{v}=2=20\log(2)\text{dB}=6\text{dB}$$

If we cascade 3 such amplifiers with different voltage gains together:



the overall voltage gain will be

$$g_{total} = 2 \times 1.4 \times 10 = 28$$

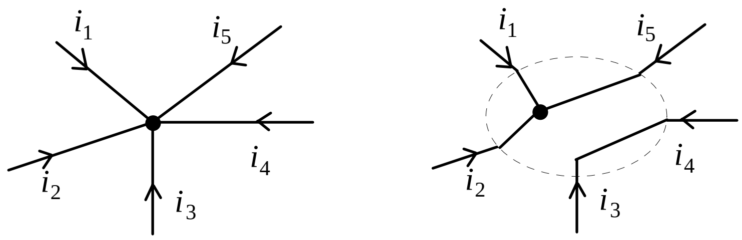
However, in dB, it is simply:

$$g_{total} = 6\,\text{dB} + 3\,\text{dB} + 20\,\text{dB} = 29\,\text{dB}$$

Under dB which is log based, multiplications become additions.

2.6 Kirchhoff's Current Law (KCL)

As demonstrated by the following examples, this states that the algebraic sum of the currents entering/leaving a node/closed surface is 0.

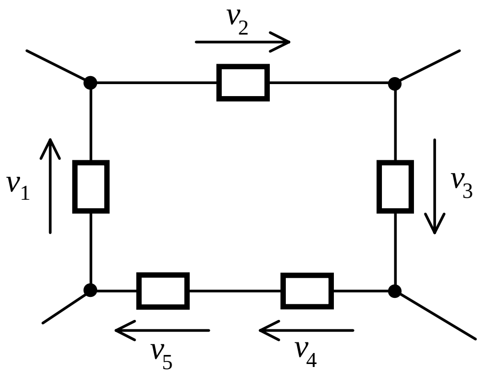


$i_1 + i_2 + i_3 + i_4 + i_5 = 0$ for both cases

Since current is equal to the rate of flow of charges, KCL actually corresponds to the conservation of charges.

2.7 Kirchhoff's Voltage Law (KVL)

As illustrated below, this states that the algebraic sum of the voltage drops around any close loop in a circuit is 0.



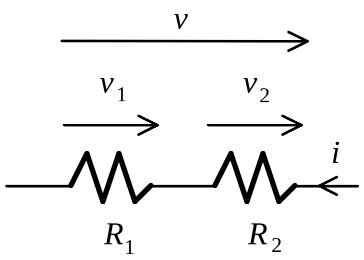
$$v_1 + v_2 + v_3 + v_4 + v_5 = 0$$

Since a charge q will have its electric potential changed by qv_1, qv_2, qv_3, qv_4, qv_5 as it passes through each of the components, the total energy change in one full loop is $q(v_1+v_2+v_3+v_4+v_5)$. Thus, from the conservation of energy:

$$v_1 + v_2 + v_3 + v_4 + v_5 = 0.$$

2.8 Series Circuit

Consider 2 resistors connected in series:



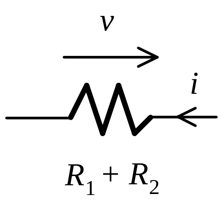
From

$$v_1 = iR_1,\ v_2 = iR_2 \text{ and } v = v_1 + v_2$$

the voltage-current relationship is

$$v = i(R_1 + R_2)$$

Now consider



The voltage/current relationship is

$$v = i(R_1 + R_2)$$

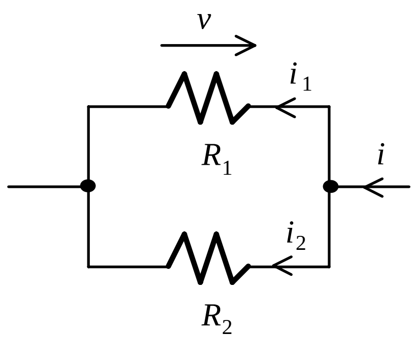
Since the voltage/current relationships are the same for both circuits, they are ***equivalent*** from an electrical point of view. In general, for n resistors $R_1, \cdots, R_n$ connected in series, the equivalent resistance R is

$$R = R_1 + \cdots + R_n$$

Clearly, the resistances of resistors connected in series add.

2.9 Parallel Circuit

Consider 2 resistors connected in parallel:



From

$$i_1 = \frac{v}{R_1},\ i_2 = \frac{v}{R_2} \text{ and } i = i_1 + i_2$$

we have

$$i = v\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Clearly, the parallel circuit is equivalent to a resistor R with voltage/current relationship

$$i = \frac{v}{R}$$

with

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

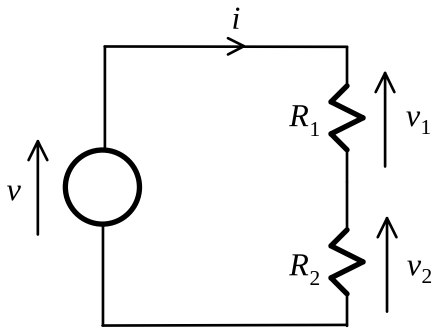
In general, for n resistors $R_1,\ \cdots,\ R_n$ connected in parallel, the equivalent resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \cdots + \frac{1}{R_n}$$

Note that $1/R$ is often called the ***conductance*** of the resistor R. Thus, the conductances of resistors connected in parallel add.

2.10 Voltage Division

Consider 2 resistors connected in series:



The total resistance of the circuit is $R_1 + R_2$. Thus:

$$i = \frac{v}{R_1 + R_2}$$

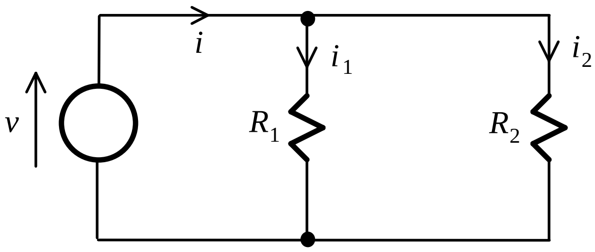
$$v_1 = iR_1 = \left(\frac{R_1}{R_1 + R_2}\right)v$$

$$v_2 = iR_2 = \left(\frac{R_2}{R_1 + R_2}\right)v$$

$$\frac{v_1}{v_2} = \frac{R_1}{R_2}$$

2.11 Current Division

Consider 2 resistors connected in parallel:



The total conductance of the circuit is

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

while the equivalent resistance is

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}}$$

Thus:

$$v = iR = \frac{i}{\frac{1}{R_1} + \frac{1}{R_2}}$$

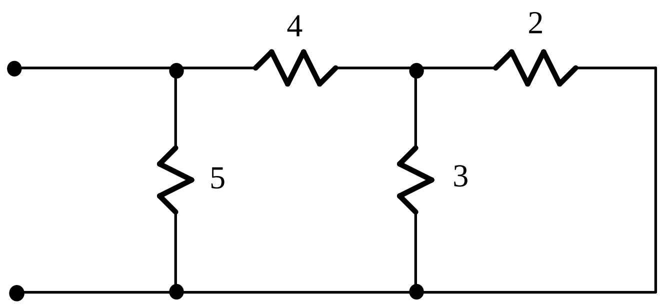
$$i_1 = \frac{v}{R_1} = \left(\frac{\frac{1}{R_1}}{\frac{1}{R_1} + \frac{1}{R_2}} \right) i = \left(\frac{R_2}{R_1 + R_2} \right) i$$

$$i_2 = \frac{v}{R_2} = \left(\frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \right) i = \left(\frac{R_1}{R_1 + R_2} \right) i$$

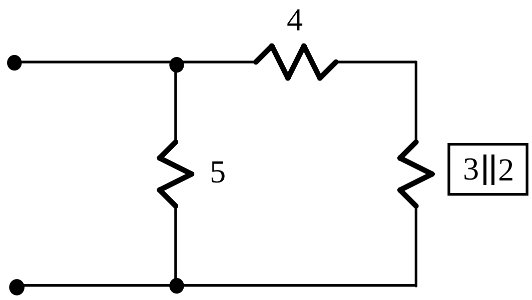
$$\frac{i_1}{i_2} = \frac{R_2}{R_1}$$

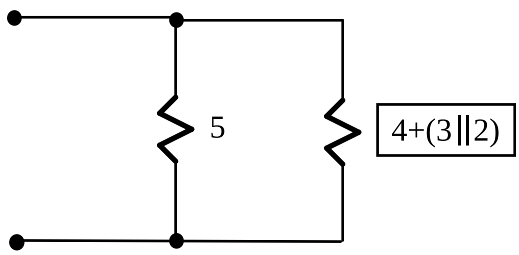
2.12 Ladder Circuit

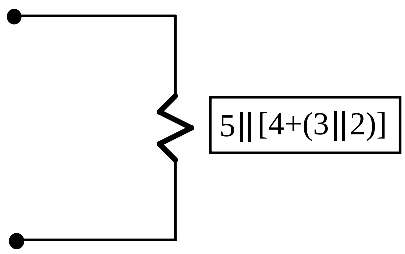
Consider the following ladder circuit:



The equivalent resistance can be determined as follows:





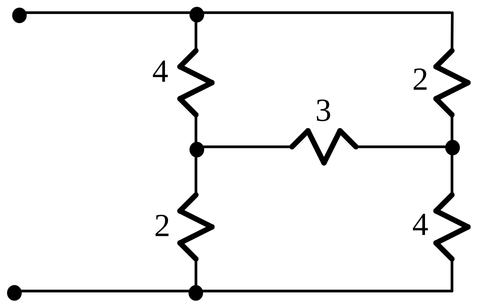


The network is equivalent to a resistor with resistance

$$R = 5 \| \left[4 + \left(3 \| 2\right)\right] = \frac{1}{\dfrac{1}{5} + \dfrac{1}{4 + \left(3 \| 2\right)}} = \frac{1}{\dfrac{1}{5} + \dfrac{1}{4 + \dfrac{1}{\dfrac{1}{3} + \dfrac{1}{2}}}}$$

2.13 Branch Current Analysis

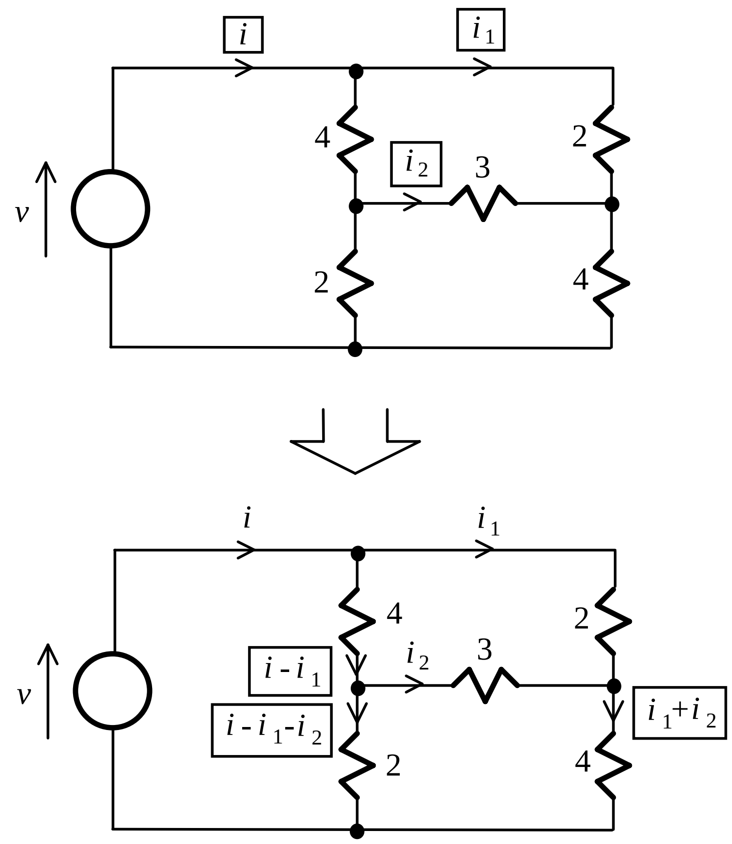
Consider the problem of determining the equivalent resistance of the following ***bridge*** circuit:



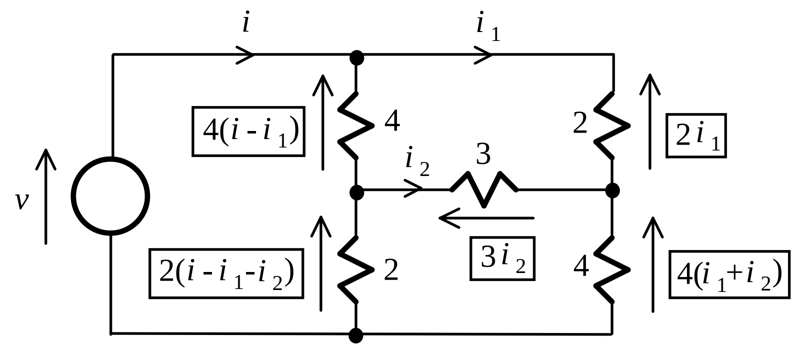
Since the components are not connected in straightforward series or parallel manner, it is not possible to use the series or parallel connection rules to simplify the circuit. However, the voltage-current relationship can be determined and this will enables the equivalent resistance to be calculated.

One method to determine the voltage-current relationship is to use the ***branch current method***.

Firstly, we arbitrarily assign 3 branch currents (so that all other branch currents can be obtained from these using KCL):



Secondly, we determine the voltage drops across the various components:



Thirdly, we apply KVL to 3 independent loops of the circuit:

$$2i_1 = 3i_2 + 4(i - i_1) \Rightarrow 6i_1 - 3i_2 = 4i$$

$$4(i_1 + i_2) + 3i_2 = 2(i - i_1 - i_2) \Rightarrow 6i_1 + 9i_2 = 2i$$

$$v = 2i_1 + 4(i_1 + i_2) = 6i_1 + 4i_2$$

Fourthly, we try to eliminate i_1 and i_2 so as to obtain the relationship between v and i :

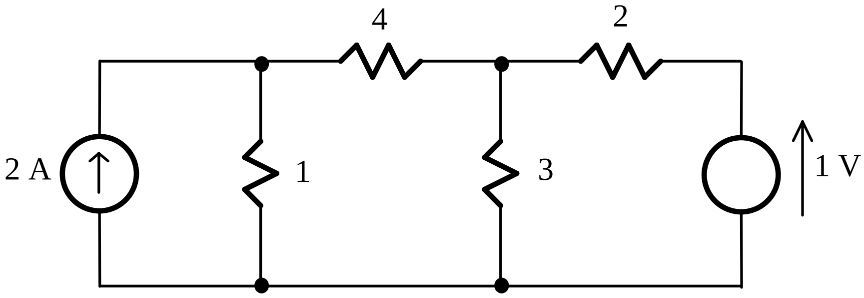
$$12i_2 = -2i \;\Rightarrow\; i_2 = -\frac{i}{6}$$

$$24i_1 = 14i \;\Rightarrow\; i_1 = \frac{7i}{12}$$

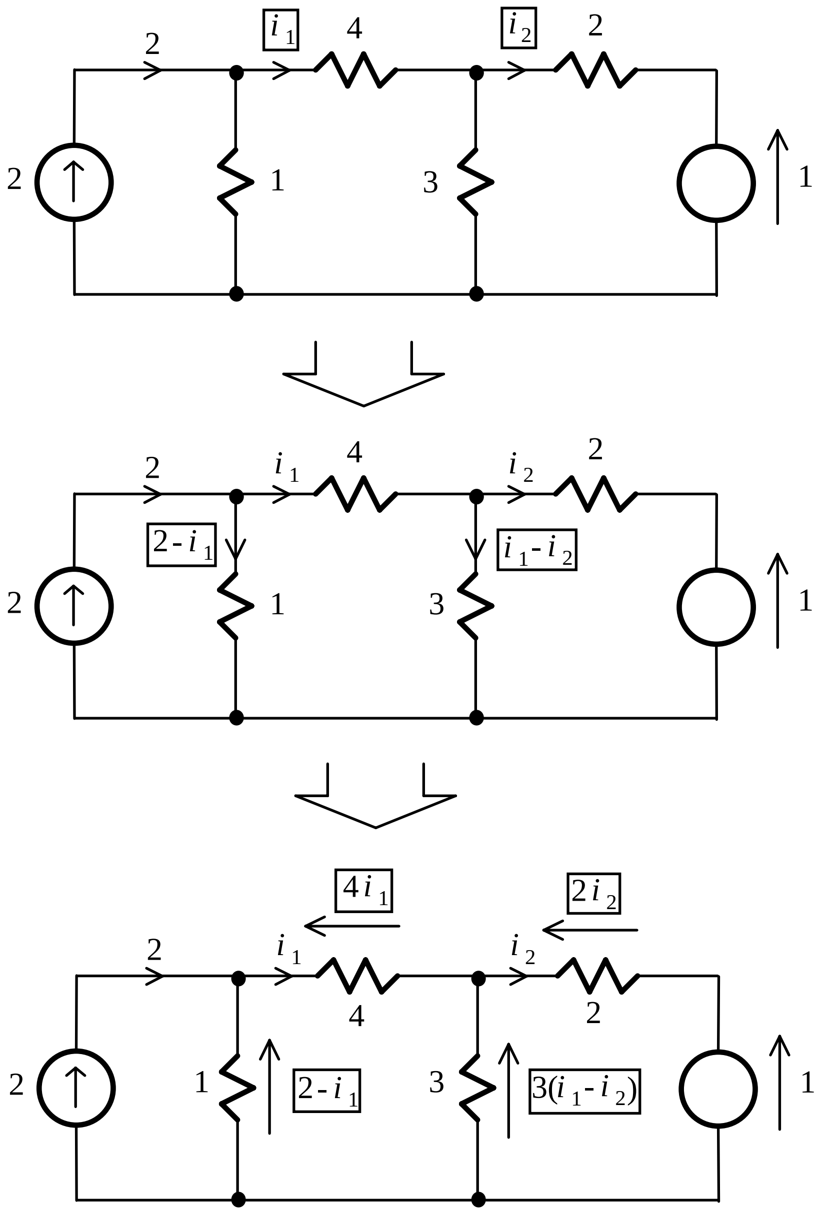
$$v = 6\left(\frac{7i}{12}\right) - 4\left(\frac{i}{6}\right) = \left(\frac{7}{2} - \frac{2}{3}\right)i = \frac{17}{6}i$$

Thus, the circuit is equivalent to a resistor of value $17/6$.

An another example, consider determining the currents/voltages in the circuit:



Again, we assign branch currents, use KCL and KVL to determine the voltage drops, and formulate the necessary equations to be solved:



$$1+2i_2=3(i_1-i_2)\Rightarrow 3i_1-5i_2=1$$

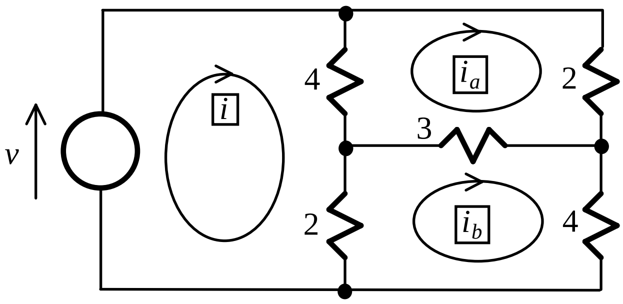
$$2-i_1=3(i_1-i_2)+4i_1\Rightarrow 8i_1-3i_2=2$$

$$\begin{bmatrix}3 & -5\\ 8 & -3\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}$$

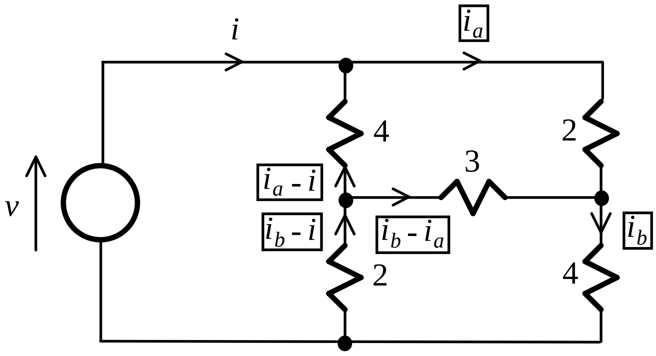
$$\begin{bmatrix}i_1\\ i_2\end{bmatrix}=\begin{bmatrix}3 & -5\\ 8 & -3\end{bmatrix}^{-1}\begin{bmatrix}1\\ 2\end{bmatrix}=\frac{1}{31}\begin{bmatrix}7\\ -2\end{bmatrix}$$

2.14 Mesh (Loop Current) Analysis

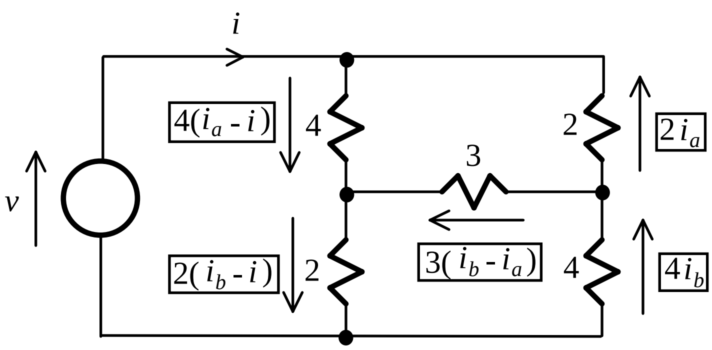
This is another common (probably more systematic) circuit analysis technique. To illustrate this method, consider the bridge circuit again. This time, we assign 3 fictitious loop currents:



We determine the branch currents:



We calculate the branch voltages:



Applying KVL and simplifying:

$$v = 2i_a + 4i_b$$

$$4(i_a - i) - 3(i_b - i_a) + 2i_a = 0 \Rightarrow 9i_a - 3i_b = 4i$$

$$2(i_b - i) + 3(i_b - i_a) + 4i_b = 0 \Rightarrow - \quad 3i_a + 9i_b = 2i$$

Expressing i_a and i_b in terms of i :

$$24i_a = 14i \Rightarrow i_a = \frac{7i}{12}$$

$$24i_b = 10i \Rightarrow i_b = \frac{5i}{12}$$

Substituting into the equation involving v:

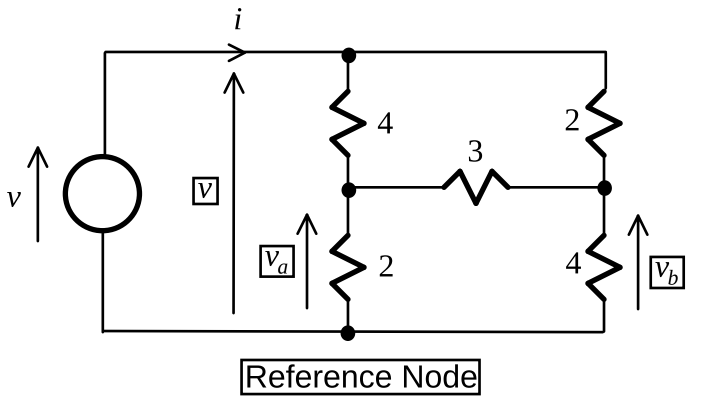
$$v = 2i_a + 4i_b = 2\left(\frac{7i}{12}\right) + 4\left(\frac{5i}{12}\right) = \frac{17i}{6}$$

The equivalent resistance is $17/6$ again.

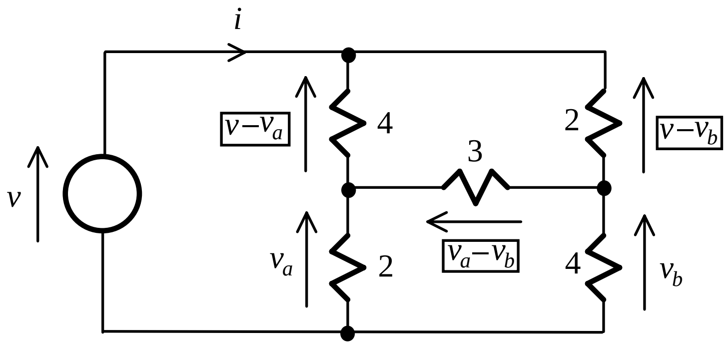
2.15 Nodal Analysis

This is probably the most systematic circuit analysis technique based on KVL and KCL. To illustrate this method, consider the bridge circuit again.

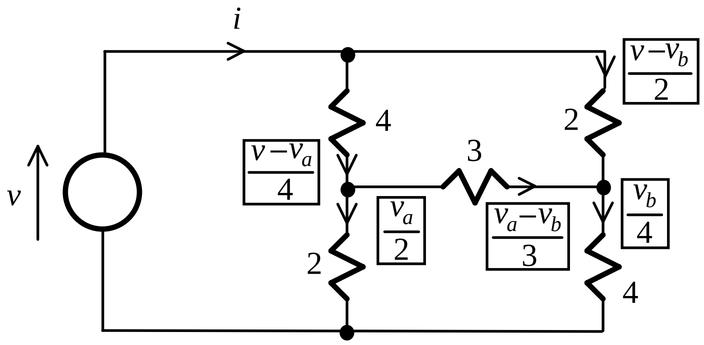
Firstly, we choose a reference node and assign voltages to other nodes with respect to this reference node:



Secondly, we determine the voltage drops across the various branches:



Thirdly, we calculate the branch currents:



Fourthly, we apply KCL:

$$i = \frac{v-v_a}{4} + \frac{v-v_b}{2} \Rightarrow i = \frac{3v}{4} - \frac{v_a}{4} - \frac{v_b}{2}$$

$$\frac{v - v_a}{4} = \frac{v_a}{2} + \frac{v_a - v_b}{3} \;\Rightarrow\; 13v_a - 4v_b = 3v$$

$$\frac{v - v_b}{2} + \frac{v_a - v_b}{3} = \frac{v_b}{4} \;\Rightarrow\; 4v_a - 13v_b = -6v$$

Expressing v_a and v_b in terms of v:

$$(13 \times 13 - 4 \times 4)v_a = (13 \times 3 + 4 \times 6)v \Rightarrow v_a = \frac{63v}{153} = \frac{7v}{17}$$

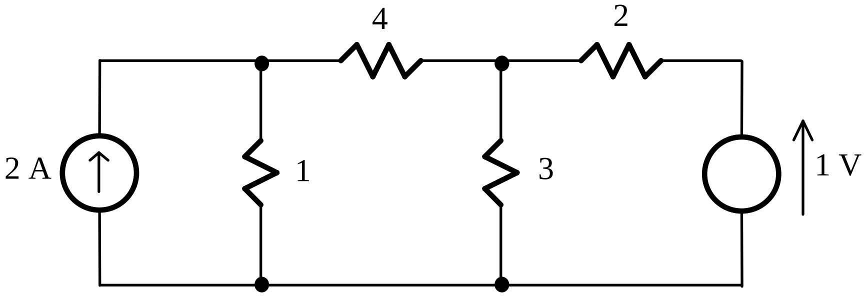
$$(13 \times 13 - 4 \times 4)v_b = (4 \times 3 + 13 \times 6)v \Rightarrow v_b = \frac{90v}{153} = \frac{10v}{17}$$

Substituting into the equation involving i:

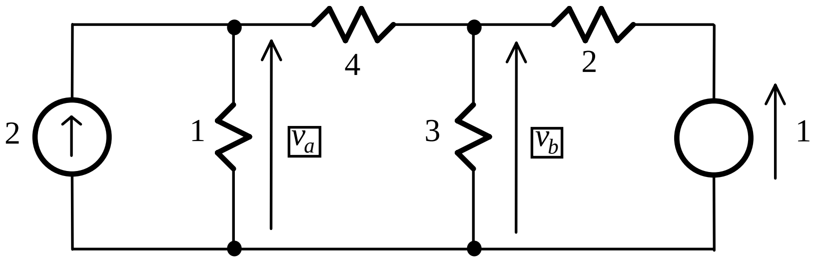
$$i = \frac{3v}{4} - \frac{v_a}{4} - \frac{v_b}{2} = \frac{3v}{4} - \frac{27v}{68} = \frac{24v}{68} = \frac{6v}{17}$$

This implies that the equivalent resistance is $17/6$.

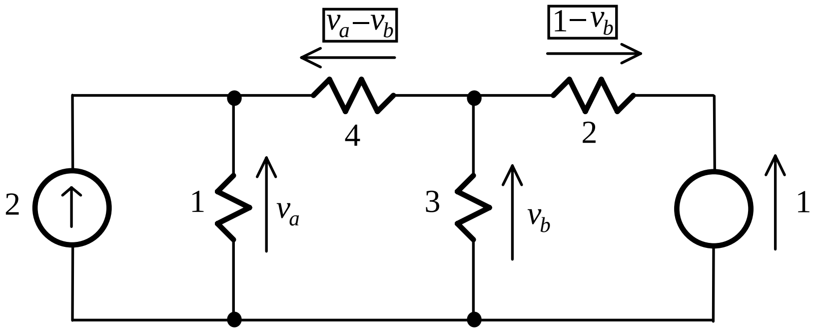
As another example, consider determining the currents/voltages in the circuit:



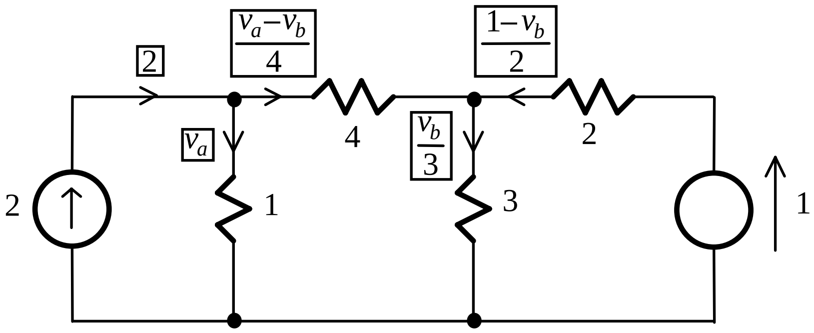
Assigning nodal voltages:



Determining the other branch voltages:



Calculating branch currents:



Applying KCL:

$$2 = v_a + \frac{v_a - v_b}{4} \;\Rightarrow\; 5v_a - v_b = 8$$

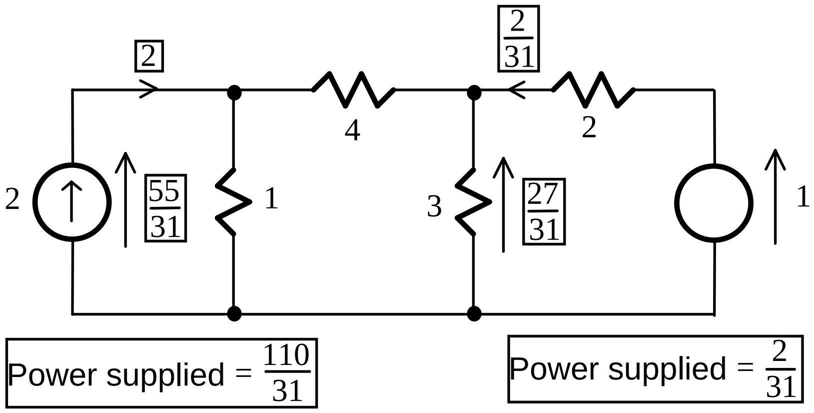
$$\frac{v_a - v_b}{4} + \frac{1 - v_b}{2} = \frac{v_b}{3} \;\Rightarrow\; 3v_a - 13v_b = -6$$

Solving:

$$v_a = \frac{55}{31}$$

$$v_b = \frac{27}{31}$$

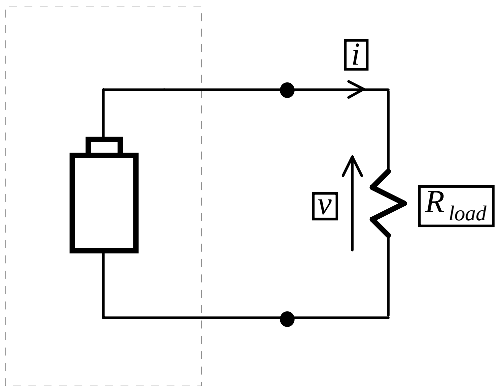
The powers supplied by the sources are:


2
2/31
4
2
2
55/31
1
3
27/31
1
Power supplied = 110/31
Power supplied = 2/31

2.16 Practical Voltage Source

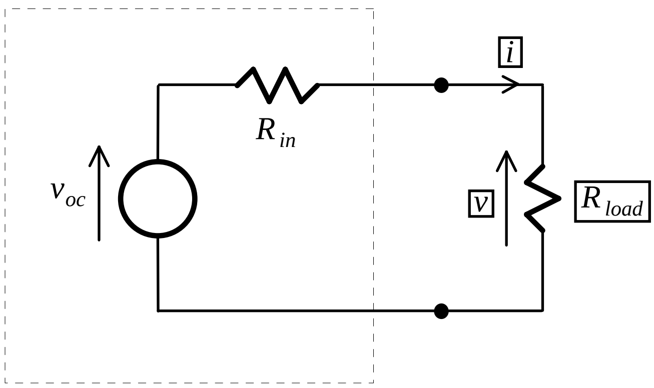
An ideal voltage source is one whose terminal voltage does not change with the current drawn. However, the terminal voltages of practical sources usually decrease slightly as the currents drawn are increased.

A commonly used model for a practical voltage source is:



Practical voltage source

|||

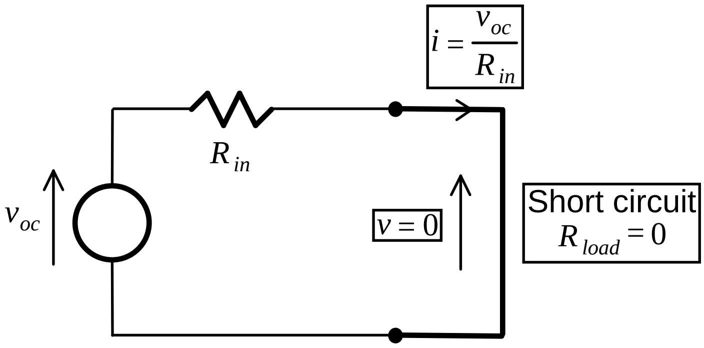


Model for voltage source

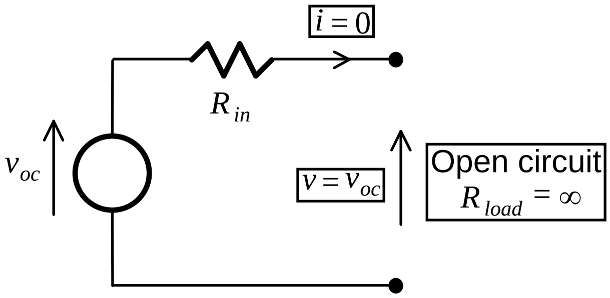
The relationship between the load current i and load voltage v is:

$$v_{oc} = v + iR_{in}$$

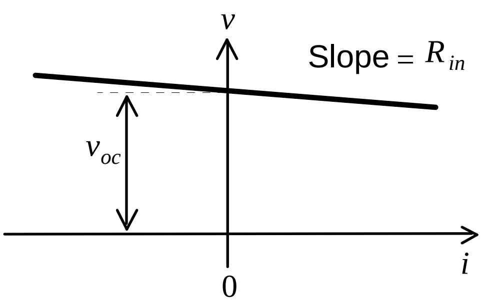
When $R_{load} = 0$ or when the source is ***short circuited*** so that $v = 0$:



When $R_{load} = \infty$ or when the source is ***open circuited*** so that $i = 0$:



Graphically:

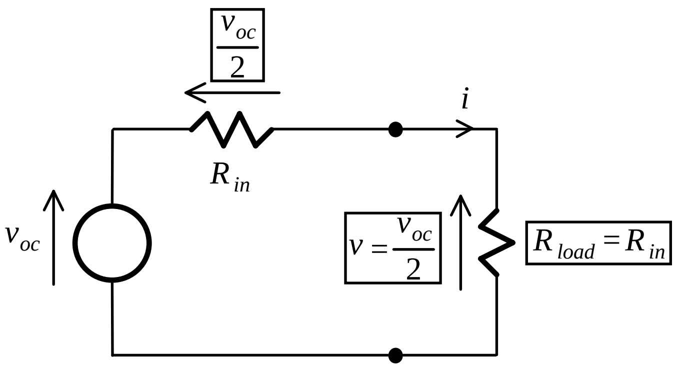


Good practical voltage source should therefore have small ***internal resistance***, R_{in}, so that its output voltage, v, will not deviate very much from v_{oc}, the ***open circuit voltage***, under any operating condition.

The internal resistance of an ideal voltage source is therefore zero so that v does not change with i .

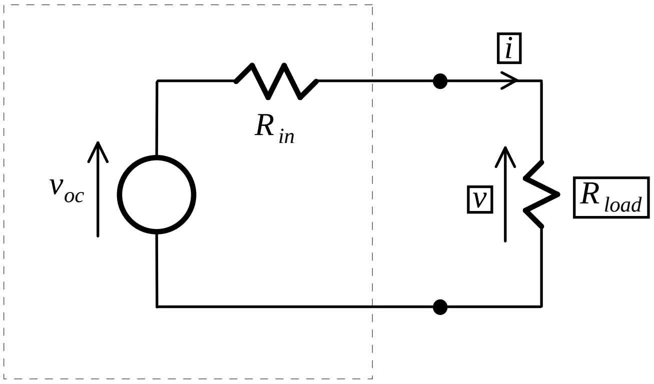
To determine the two parameters v_{oc} and R_{in} that characterise, say, a battery, we can measure the output voltage when the battery is open-circuited (nothing connected except the voltmeter). This will give v_{oc}.

Next, we can connect a load resistor and vary the load resistor such that the voltage across it is $v_{oc}/2$. The load resistor is then equal to R_{in}:


$\frac{v_{oc}}{2}$
i
R_{in}
v_{oc}
$v = \frac{v_{oc}}{2}$
$R_{load} = R_{in}$

2.17 Maximum Power Transfer

Consider the following circuit:



Model for voltage source

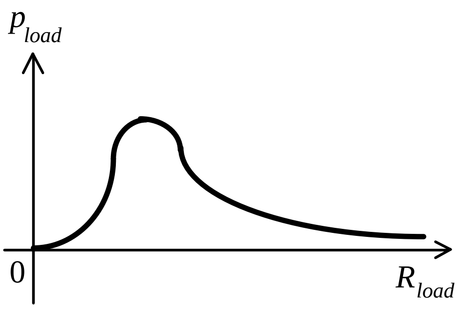
The current in the load resistor is

$$i = \frac{v_{oc}}{R_{in} + R_{load}}$$

The power absorbed by the load resistor is

$$p_{load} = i^2 R_{load} = \frac{v_{oc}^{\ 2} R_{load}}{\left(R_{in} + R_{load}\right)^2}$$

This is always positive. However, if $R_{load} = 0$ or $R_{load} = \infty$, $p_{load} = 0$. Thus:



Differentiating:

$$\frac{dp_{load}}{dR_{load}} = v_{oc}^{\ 2}\left[\frac{1}{\left(R_{in}+R_{load}\right)^2} - \frac{2R_{load}}{\left(R_{in}+R_{load}\right)^3}\right] = v_{oc}^{\ 2}\left[\frac{R_{in}-R_{load}}{\left(R_{in}+R_{load}\right)^3}\right]$$

The load resistor will be absorbing the maximum power or the source will be transferring the maximum power if

$$R_{in} = R_{load}$$

or when the load and source internal resistances are ***matched***.

Note that maximum power transfer may not be always desirable, although it is commonly used for, say, driving hi fi speakers.

This is because the power absorbed by the source internal resistance is

$$p_{in} = i^2 R_{in} = \frac{{v_{oc}}^2 R_{in}}{\left(R_{in} + R_{load}\right)^2}$$

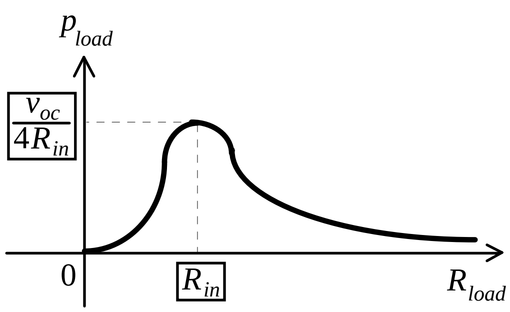
and this increases as i or more current is drawn. Practically, this power is wasted and the source will heat up as i increases.

To waste as little power as possible, it is desirable that i be as small as possible. Thus, in transmitting power to homes and industries, high voltages (kV) are used so that for the same power, the current drawn is as small as possible. The transmitted high tension voltages are transformed to lower values ($230\ \mathrm{V}$) by transformers located nearer to the customers.

The maximum power transfer formula also gives the maximum amount of power that a practical source is capable of giving. With $R_{load} = R_{in}$, this is

$$p_{load} = \frac{{v_{oc}}^2}{4R_{in}}$$

Graphically:



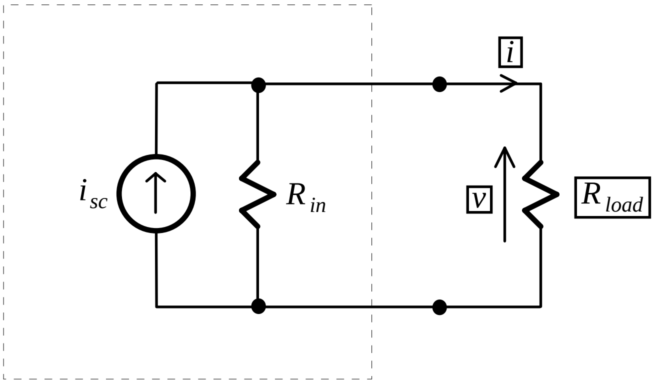
The smaller the internal resistance, the larger the power the source is capable of delivering. High power source, say, car battery, therefore has very little internal resistance. If accidentally shorted by a piece of wire, the wire may be "melted" or the battery may be damaged due to the heat given out.

A voltage source with zero resistance, an ideal voltage source, is therefore an idealisation in that it is capable to deliver an infinite amount of power.

2.18 Practical Current Source

An ideal current source is one which delivers a constant current regardless of its terminal voltage. However, the current delivered by a practical current source usually changes slightly depending on the load and the terminal voltage.

A commonly used model for a current source is:

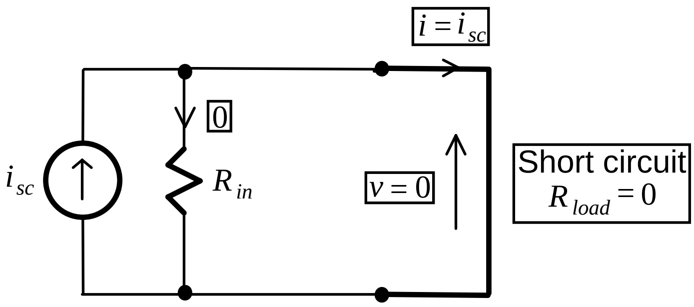


Model for current source

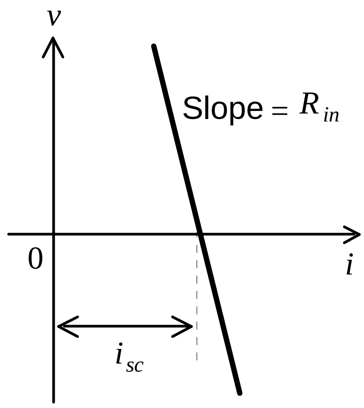
The relationship between the load current i and load voltage v is:

$$i_{sc} = \frac{v}{R_{in}} + i$$

When $R_{load} = 0$ or when the source is short-circuited so that $v = 0$:



Graphically:

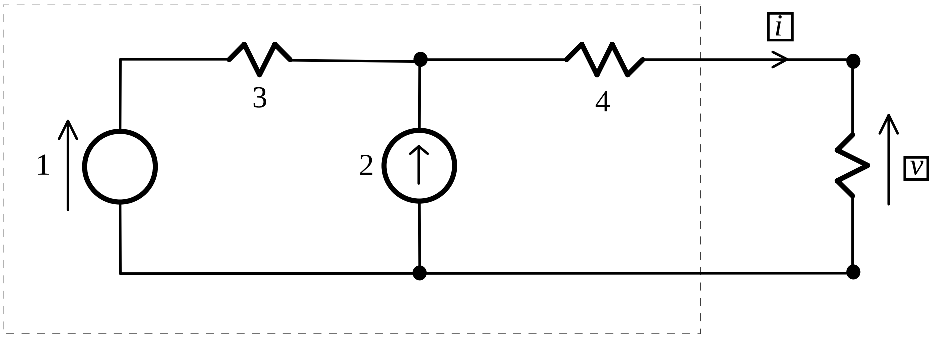


Good practical current source should therefore have large internal resistance R_{in} so that the current i delivered does not deviate very much from i_{sc}, the ***short circuit current***, under any operating condition.

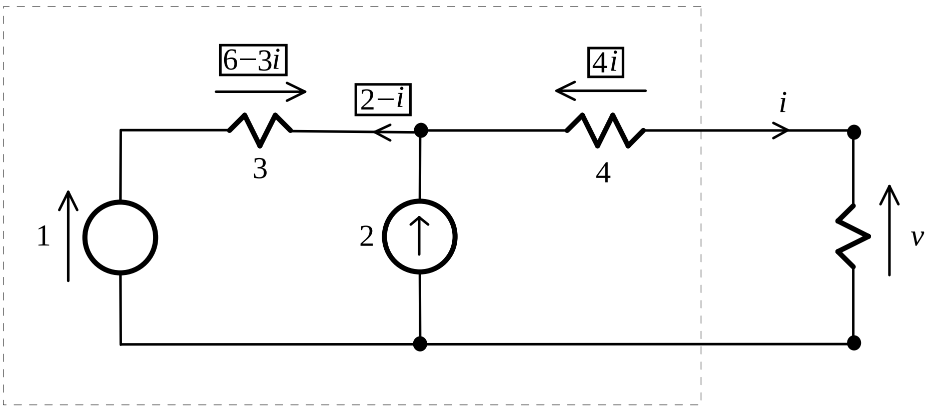
The internal resistance of an ideal current source is therefore infinity so that i does not change with v.

2.19 Thevenin's Equivalent Circuit

Consider the circuit



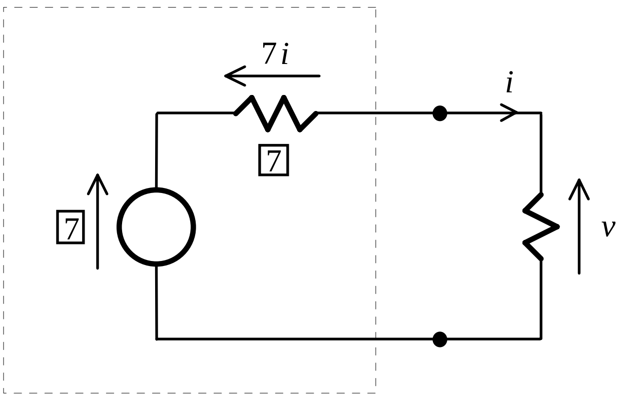
To find the relationship between i and v:



Applying KVL:

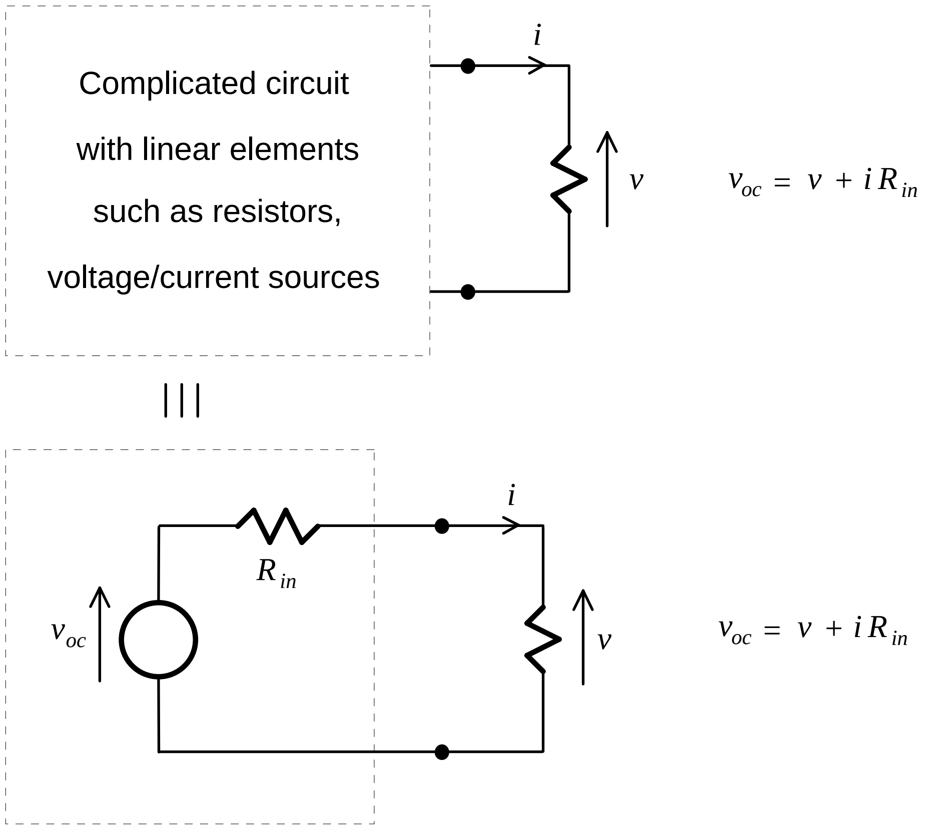
$$1+(6-3i)=4i+v \Rightarrow 7=v+7i$$

The circuit is therefore equivalent to:



which has the same voltage-current relationship of $7 = v + 7i$.

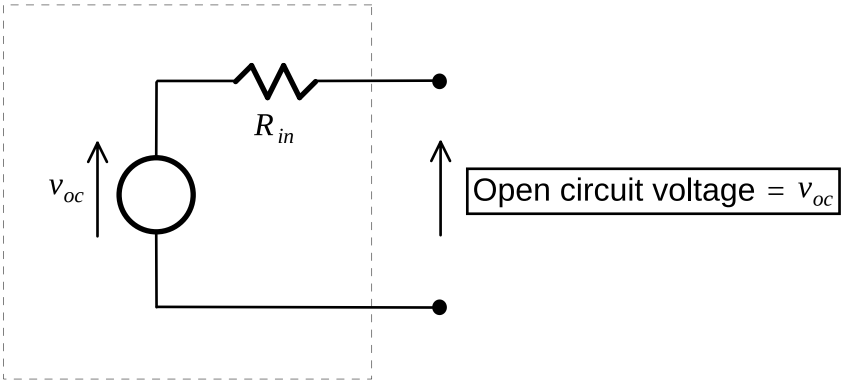
In general, since the voltage-current relationships of ideal resistors, voltage and current sources are ***linear*** in nature, any circuit comprising these elements will also be linear, has linear voltage-current relationship and will be equivalent to an ideal voltage source in series with a resistor, the ***Thevenin's equivalent circuit***:

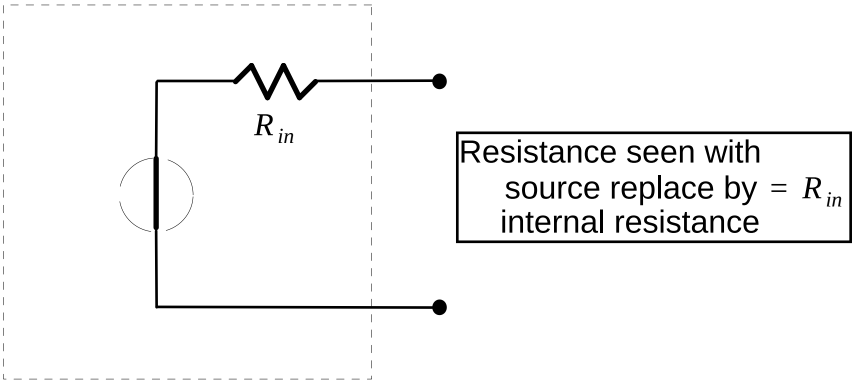


Thevenin's equivalent circuit

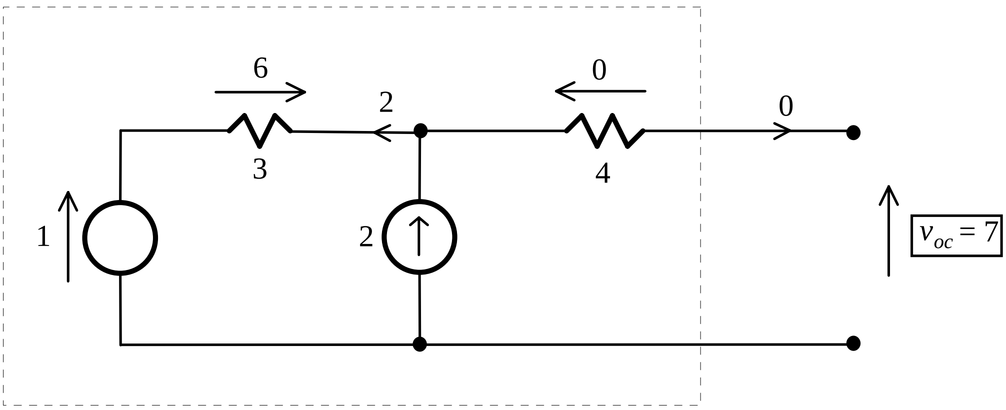
The values of v_{oc} and R_{in} can be found by determining the relationship between v and i from using KCL and KVL on the original circuit as demonstrated in the previous example.

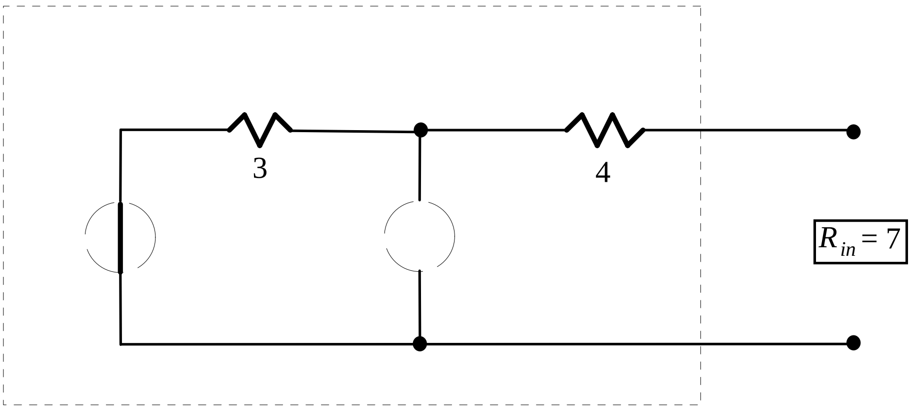
Alternatively, note that from the Thevenin's equivalent circuit:





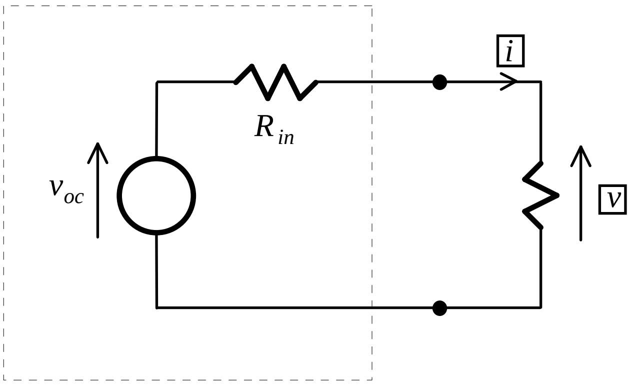
Thus, v_{oc} and R_{in} for the circuit analysed can also be calculated in the same manner:





2.20 Norton's Equivalent Circuit

Another commonly used equivalent circuit is the Norton's equivalent circuit, which can be considered to be the dual of the Thevenin's equivalent circuit:

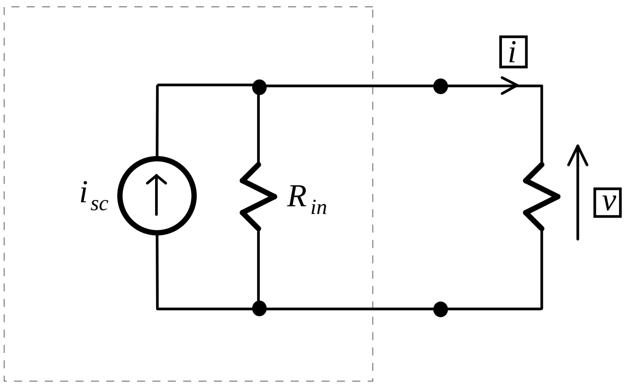


$$v_{oc} = v + i R_{in}$$

Thevenin's equivalent circuit

||| if $v_{oc} = i_{sc} R_{in}$

||| if $v_{oc} = i_{sc} R_{in}$

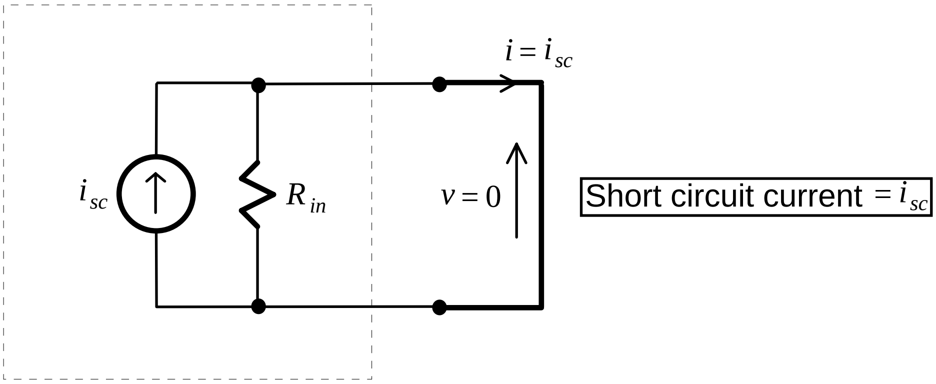


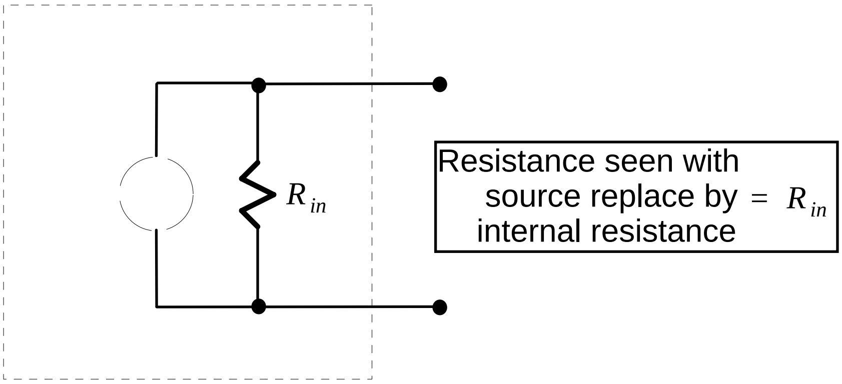
$$i_{sc} = \frac{v}{R_{in}} + i$$

$$i_{sc} R_{in} = v + i R_{in}$$

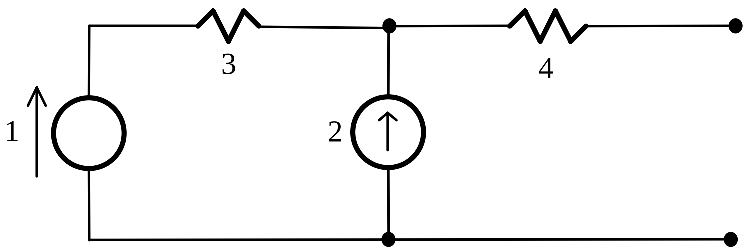
Norton's equivalent circuit

From the Norton's equivalent circuit, the two parameters i_{sc} and R_{in} can be obtained from:

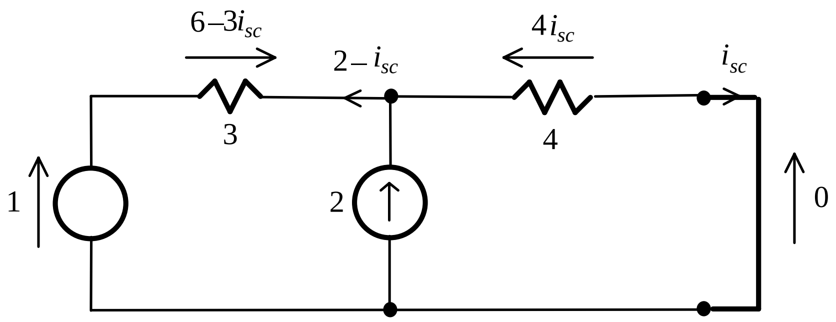




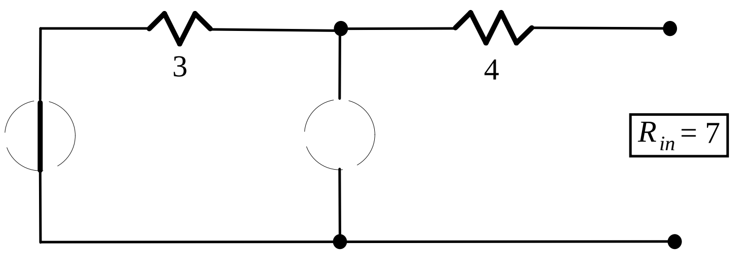
Thus, the determination of the Norton's equivalent circuit for



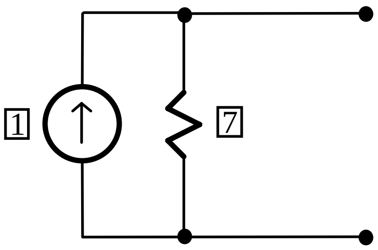
can proceed as follows:



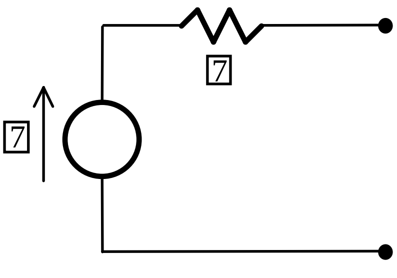
$1+6-3i_{sc} = 4i_{sc}$ or $i_{sc} = 1$



The Norton's equivalent circuit is therefore:

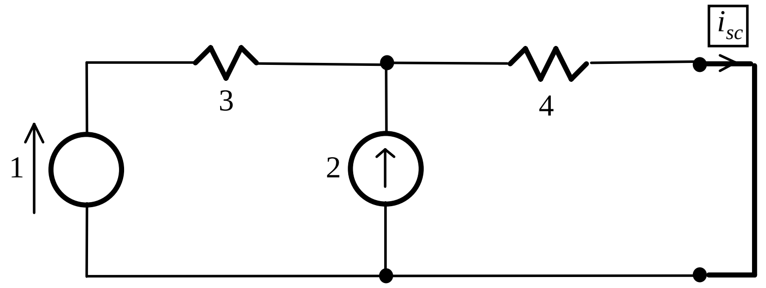


And the Thevenin's equivalent circuit is:

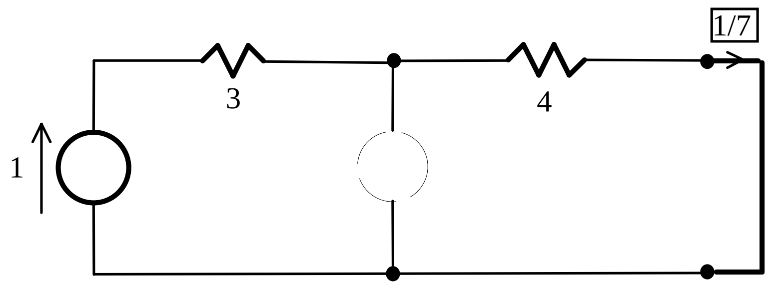


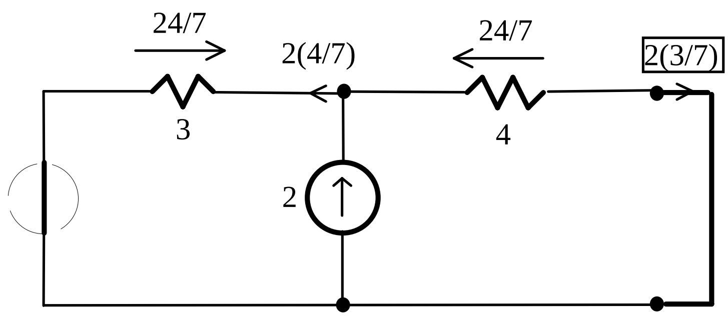
2.21 Superposition

Consider finding i_{sc} in the circuit:

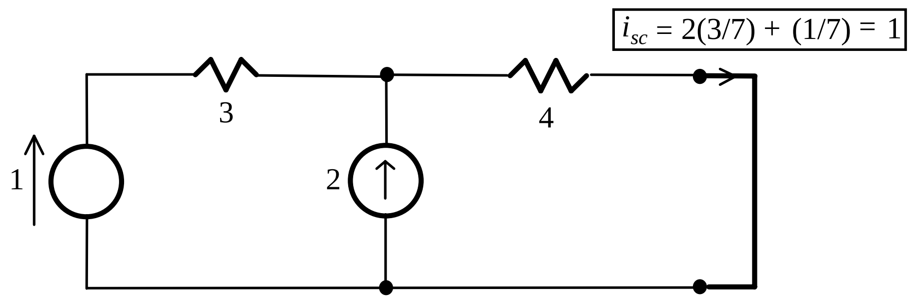


By using the principle of superposition, this can be done by finding the components of i_{sc} due to the 2 ***independent*** sources on their own (with the other sources replaced by their internal resistances):





The actual value of i_{sc} when both sources are present is given by the sum of these components:



Basically, the principle of superposition works because the circuit is linear in nature. For these circuits, the ***output/response*** is related to the ***inputs/excitations*** by a linear equation such as

$$\text{output} = k_1(\text{input}_1) + k_2(\text{input}_2)$$

Therefore, if output_1 is the output by using input_1 on its own with input_2 removed so that:

$$\text{output}_1 = k_1(\text{input}_1) + k_2(0) = k_1(\text{input}_1)$$

while output_2 is the output by using input_2 on its own with input_1 removed so that:

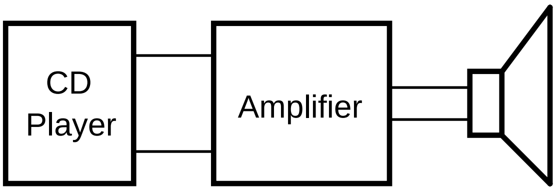
$$\text{output}_2 = k_1(0) + k_2(\text{input}_2) = k_2(\text{input}_2)$$

then the actual output with both inputs present will be

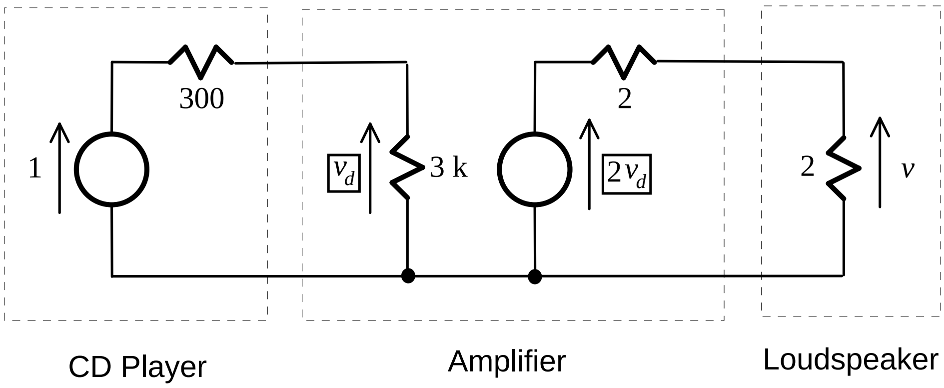
$$\text{output} = \text{output}_1 + \text{output}_2$$

2.22 Dependent Source

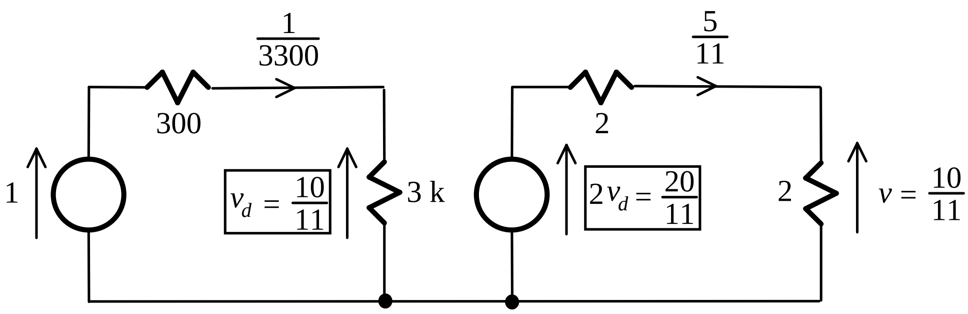
Consider the following system:



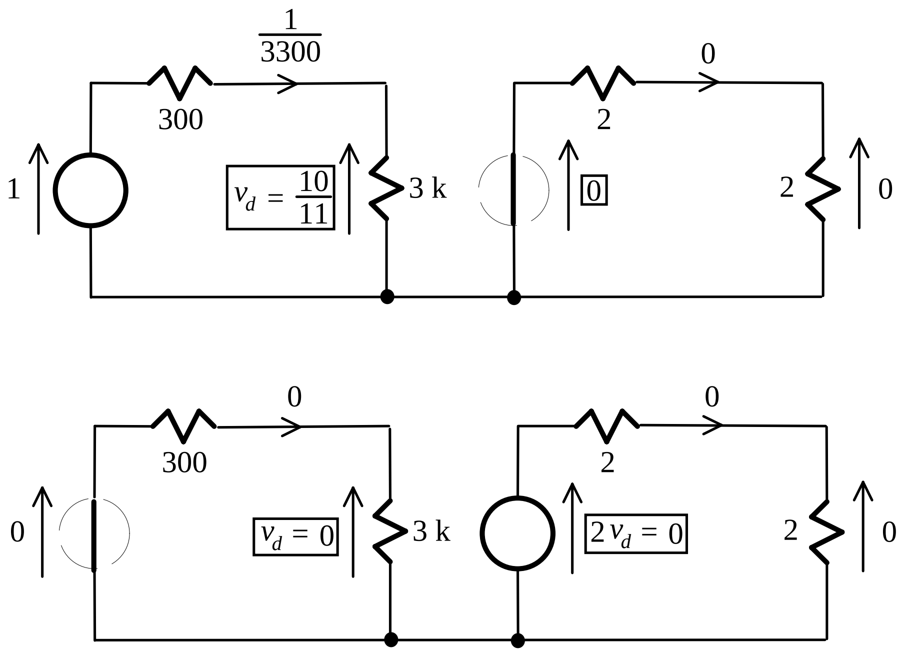
This may be represented by



Note that the source in the Amplifier block is a ***dependent source***. Its value depends on v_d, the voltage across the inputs of the amplifier. Using KCL and KVL, the voltage v can be easily found:



However, if we use the principle of superposition treating the dependent source as an independent source (which is wrong !), the value of v will be 0:



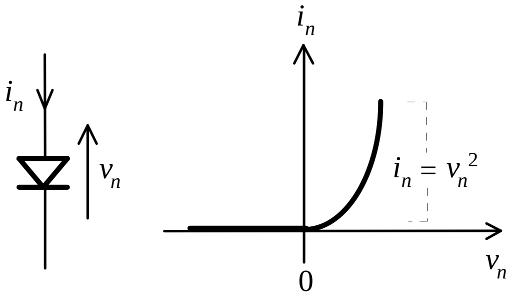
Dependent sources, which depend on other voltages/currents in the circuit and are therefore not independent excitations, cannot be removed when the principle of superposition is used. They should be treated like other passive components such as resistors in circuit analysis.

2.23 Non-Linear Circuit

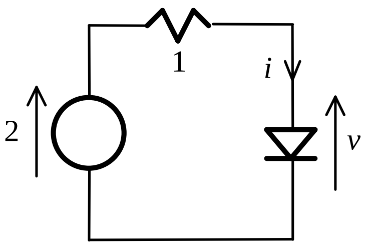
The circuits analysed so far are linear in nature and a large number of techniques exist for analysing these circuits. Non-linear circuits, with non-linear circuit elements, are much more difficult to be analysed.

The currents and voltages in non-linear circuits are usually found from solving, usually using computers, the voltage-current relationships governing the non-linear element components and those obtained based on KVL and KCL.

Consider using the following non-linear device



in the circuit



The voltage v and current i can be determined by using the voltage-current relationship for the device and KVL and solving the resulting non-linear equations:

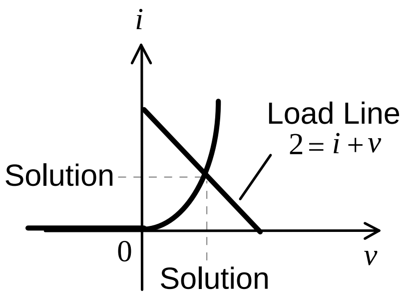
$$2 = i + v$$

$$i = f(v) = \begin{cases} v^2, & v \geq 0 \\ 0, & v < 0 \end{cases}$$

which give

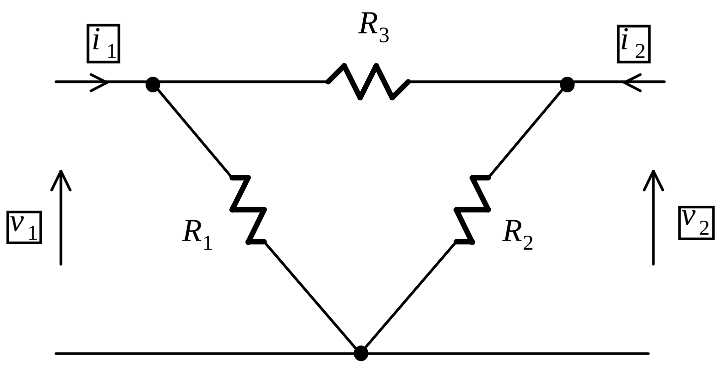
$$2 = v + v^2 \Rightarrow v = 1$$

Graphically:

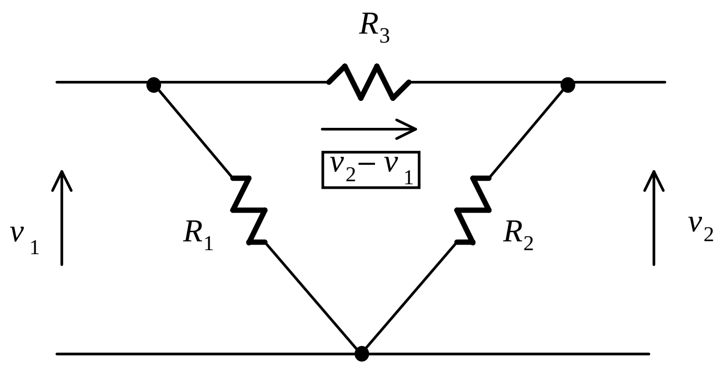

i
Load Line
2 = i + v
Solution
0
v
Solution

2.24 Delta Circuit

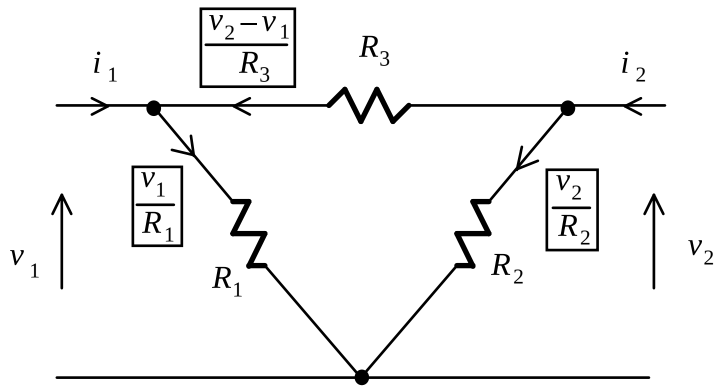
Consider the following delta circuit:



Given v_1 and v_2, the voltages across the various resistors are:



The currents through the various resistors are:



From KCL, the currents i_1 and i_2 are related to the voltages applied by

$$i_1 = \frac{v_1}{R_1} - \frac{v_2 - v_1}{R_3} = \left(\frac{1}{R_1} + \frac{1}{R_3}\right) v_1 - \frac{v_2}{R_3}$$

$$i_2 = \frac{v_2}{R_2} + \frac{v_2 - v_1}{R_3} = -\frac{v_1}{R_3} + \left(\frac{1}{R_2} + \frac{1}{R_3}\right) v_2$$

The delta circuit is a ***2-port 3-terminal network***, whose electrical characteristic is given by 2 equations relating the 2 applied voltages and the 2 resulting currents.

For comparison, a resistor is a 1-port network whose electrical characteristic is given by a single equation, $v = iR$, on the applied voltage and the resulting current.

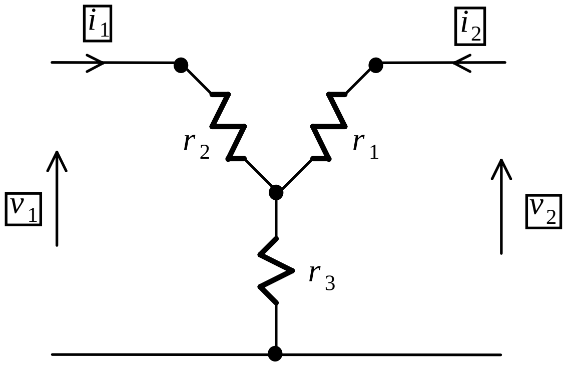
The 2 voltage-current equations for the delta circuit is often written in matrix form:

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

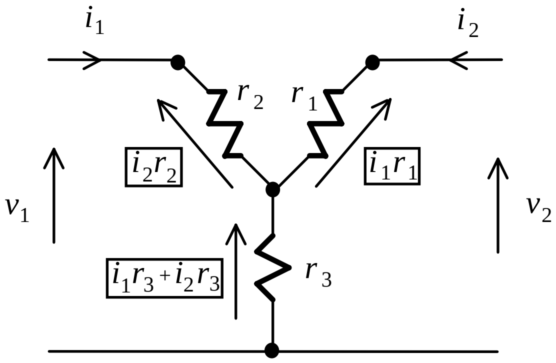
The matrix involved is called the ***conductance*** matrix.

2.25 Star Circuit

Consider the following star circuit:



Given i_1 and i_2, the voltages across the various resistors are:



The voltage-current relationships are:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} r_2 i_1 + r_3(i_1 + i_2) \\ r_1 i_2 + r_3(i_1 + i_2) \end{bmatrix} = \begin{bmatrix} (r_2 + r_3) i_1 + r_3 i_2 \\ r_3 i_1 + (r_1 + r_3) i_2 \end{bmatrix} = \begin{bmatrix} r_2 + r_3 & r_3 \\ r_3 & r_1 + r_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

or

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} r_2 + r_3 & r_3 \\ r_3 & r_1 + r_3 \end{bmatrix}^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{\begin{bmatrix} r_1 + r_3 & -r_3 \\ -r_3 & r_2 + r_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}{(r_2 + r_3)(r_1 + r_3) - r_3^2} = \frac{\begin{bmatrix} r_1 + r_3 & -r_3 \\ -r_3 & r_2 + r_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}{r_1 r_2 + r_2 r_3 + r_3 r_1}$$

Comparing this with

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_3} & -\frac{1}{R_3} \\ -\frac{1}{R_3} & \frac{1}{R_2} + \frac{1}{R_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

for the star circuit, the star and delta circuits are equivalent if

$$\frac{1}{R_3} = \frac{r_3}{r_1 r_2 + r_2 r_3 + r_3 r_1}$$

$$\frac{1}{R_1} = \frac{r_1}{r_1 r_2 + r_2 r_3 + r_3 r_1}$$

$$\frac{1}{R_2} = \frac{r_2}{r_1 r_2 + r_2 r_3 + r_3 r_1}$$

This is called the ***star-to-delta/delta-to-star transformation***.

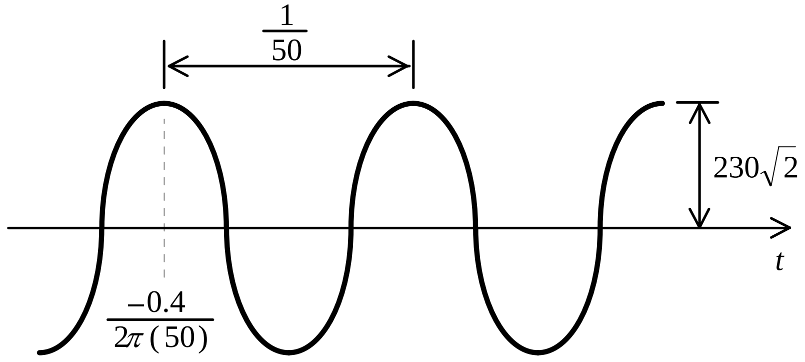
3

AC CIRCUIT ANALYSIS

3.1 Sources

In dc circuits, the voltages and currents are constants and do not change with time. In ac (***alternating current***) circuits, the voltages and currents change with time in a sinusoidal manner.

The most common ac voltage source is the mains:



Mathematically:

$$v(t) = \sqrt{2}r\cos(2\pi ft + \theta) = \sqrt{2}r\cos(\omega t + \theta) = \sqrt{2}r\cos\left(\frac{2\pi t}{T} + \theta\right)$$

$$= 230\sqrt{2}\cos(100\pi t + 0.4)$$

$\theta =$ ***phase*** $= 0.4$ rad

$f =$ ***frequency*** $= 50$ Hz

$\omega = 2\pi f =$ ***angular frequency*** $= 100\pi = 314$ rad/s

$T = \frac{1}{f} =$ ***period*** $= \frac{1}{50} = 0.02$ s

$\sqrt{2}r =$ ***peak value*** $= 230\sqrt{2} = 324$ V

$r =$ ***rms (root mean square) value*** $= 230$ V

Most ac voltage/current values are rms based as it is more convenient to use these values in the calculation of powers.

3.2 Phasor

A sinusoidal voltage/current is usually represented using complex number format:

$$v(t) = \sqrt{2}r\cos(\omega t + \theta) = \sqrt{2}r\,\mathrm{Re}\left[e^{j(\omega t+\theta)}\right] = \mathrm{Re}\left[\left(re^{j\theta}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

The advantage of this can be seen if, say, we have to add 2 sinusoidal voltages given by:

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) \text{ and } v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

The addition can proceed as follows:

$$v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$v_2(t) = 5\sqrt{2}\cos\left(\omega - \frac{\pi}{4}\right) = \mathrm{Re}\left[\left(5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$v_1(t) + v_2(t) = \mathrm{Re}\left[\left(3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}}\right)\left(\sqrt{2}e^{\omega t}\right)\right] = \mathrm{Re}\left[\left(6.47e^{-j0.32}\right)\left(\sqrt{2}e^{j\omega t}\right)\right]$$

$$= 6.47\sqrt{2}\cos(\omega t - 0.32)$$

Note that the complex time factor $\sqrt{2}e^{j\omega t}$ appears in all the expressions. If we represent $v_1(t)$ and $v_2(t)$ by the complex numbers or ***phasors***:

$$V_1 = 3e^{j\frac{\pi}{6}} \quad \text{representing} \quad v_1(t) = 3\sqrt{2}\cos\left(\omega t + \frac{\pi}{6}\right)$$

$$V_2 = 5e^{-j\frac{\pi}{4}} \quad \text{representing} \quad v_2(t) = 5\sqrt{2}\cos\left(\omega t - \frac{\pi}{4}\right)$$

then the phasor representation for $v_1(t) + v_2(t)$ will be

$$V_1 + V_2 = 3e^{j\frac{\pi}{6}} + 5e^{-j\frac{\pi}{4}} = 6.47e^{-j0.32} \quad \text{representing}$$

$$v_1(t) + v_2(t) = 6.47\sqrt{2}\cos(\omega t - 0.32)$$

By using phasors, a time-varying ac voltage

$$v(t) = \sqrt{2} r \cos(\omega t + \theta) = \text{Re}\left[\left(r e^{j\theta}\right)\left(\sqrt{2} e^{j\omega t}\right)\right]$$

becomes a simple complex time-invariant number/voltage

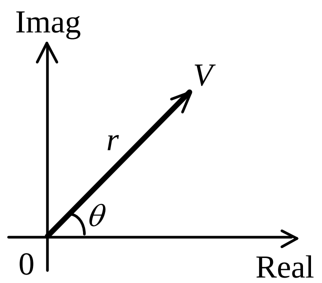
$$V = r e^{j\theta} = r\angle\theta$$

with

$r = |V| =$ magnitude/modulus of $V =$ r.m.s. value of $v(t)$

$\theta = \text{Arg}[V] =$ phase of V

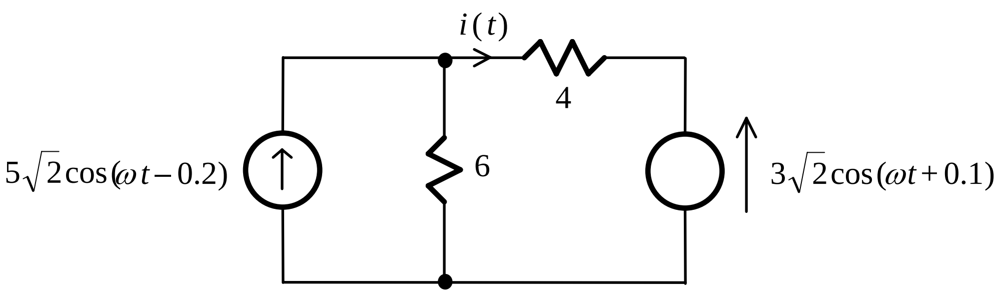
Graphically, on a phasor diagram:



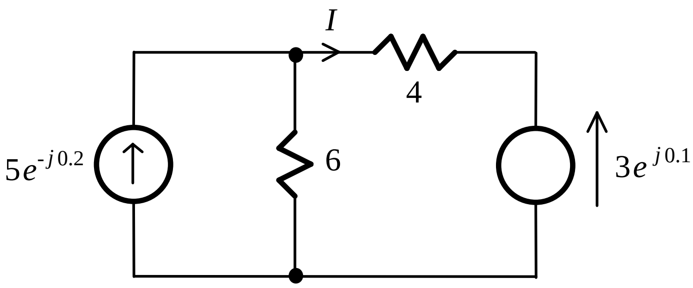
Complex Plane

Using phasors, all time-varying ac quantities become complex dc quantities and all dc circuit analysis techniques can be employed for ac circuit with virtually no modification.

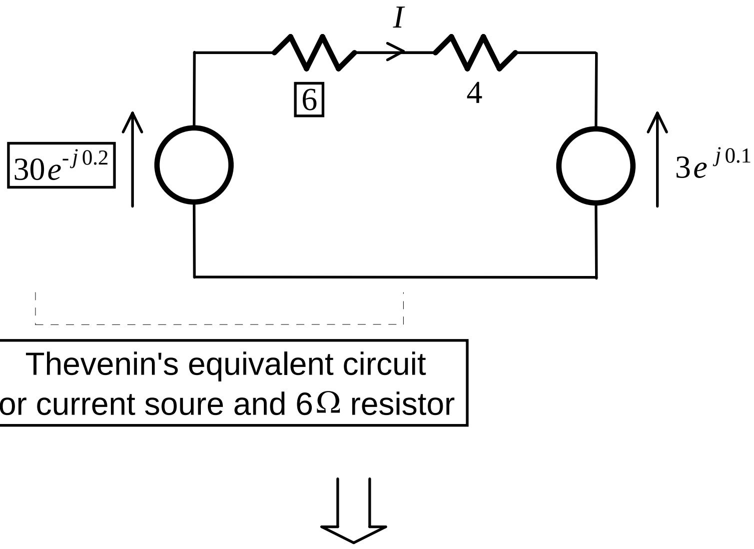
As an example, the ac circuit



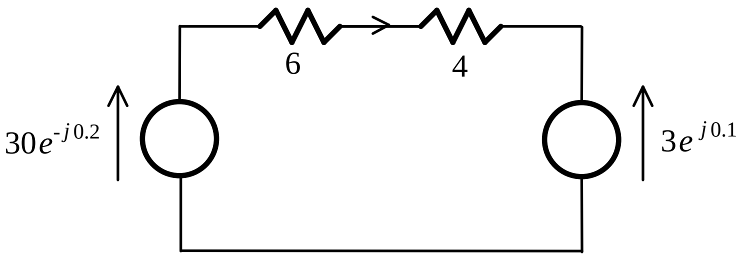
is usually represented by



and finding $i(t)$ in the actual circuit can be done by:



$$I = \frac{30e^{-j0.2} - 3e^{j0.1}}{10} = 2.64 - j0.63 = 2.71e^{-j0.23}$$



From the phasor I, $i(t)$ is:

$$i(t) = 2.71\sqrt{2}\cos(\omega t - 0.23)$$

Note that since there is a one-to-one correspondence between the actual ac voltage/current and the associated phasor, most ac circuits are described in terms of phasors. Thus, the answer to the above problem can be given in terms of $I = 2.71e^{-j0.23}$ rather than $i(t) = 2.71\sqrt{2}\cos(\omega t - 0.23)$.

3.3 Root Mean Square Value

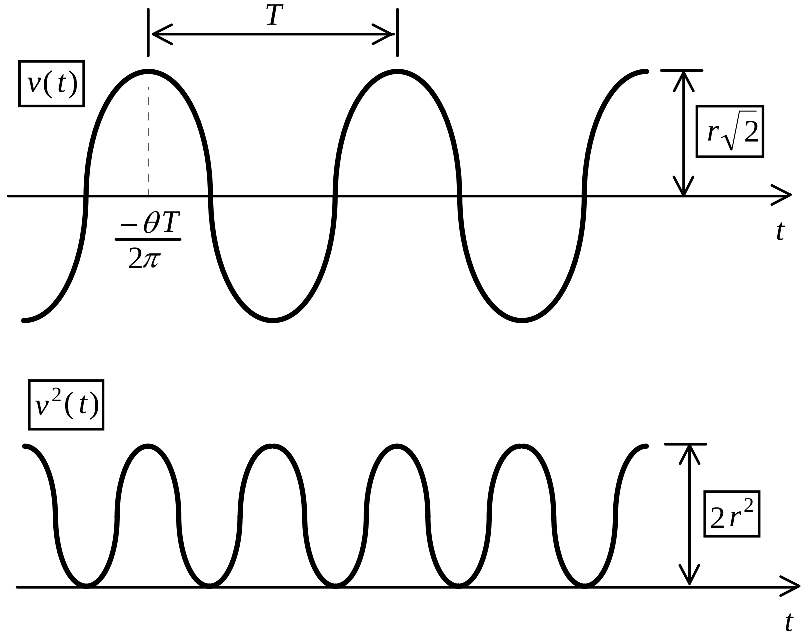
For the ac voltage

$$v(t)=\sqrt{2}r\cos(2\pi ft+\theta)=\sqrt{2}r\cos\left(\frac{2\pi t}{T}+\theta\right)$$

the square value, using $\cos(2x)=2\cos^2(x)-1$, is

$$v^2(t)=2r^2\cos^2\left(\frac{2\pi t}{T}+\theta\right)=r^2\left[1+\cos\left(\frac{4\pi t}{T}+2\theta\right)\right]$$

Graphically:



The average or mean of the square value is

$$\frac{1}{1\,\text{period}}\int_{1\,\text{period}} v^2(t)dt=\frac{1}{T}\int_0^T v^2(t)dt$$

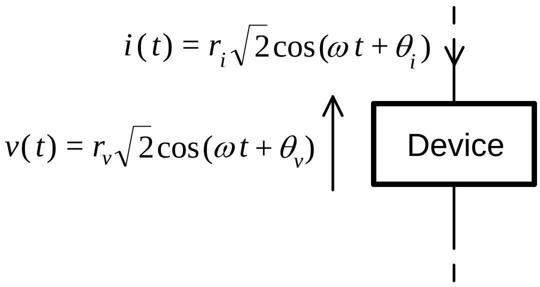
$$\frac{1}{T}\int_0^T r^2\left[1+\cos\left(\frac{4\pi t}{T}+2\theta\right)\right]dt=\frac{1}{T}\int_0^T r^2dt=r^2$$

The square root of this or the ***rms*** (***root mean square***) value of $v(t)$ is

rms value of $\sqrt{2}r\cos(\omega t+\theta)=r$

3.4 Power

Consider the ac device:



Using $2\cos(x_1)\cos(x_2) = \cos(x_1 - x_2) + \cos(x_1 + x_2)$, the ***instantaneous power*** consumed is

$$p(t) = i(t)v(t) = 2r_i r_v \cos(\omega t + \theta_i)\cos(\omega t + \theta_v)$$

$$= r_i r_v\left[\cos(\theta_i - \theta_v) + \cos(2\omega t + \theta_i + \theta_v)\right]$$

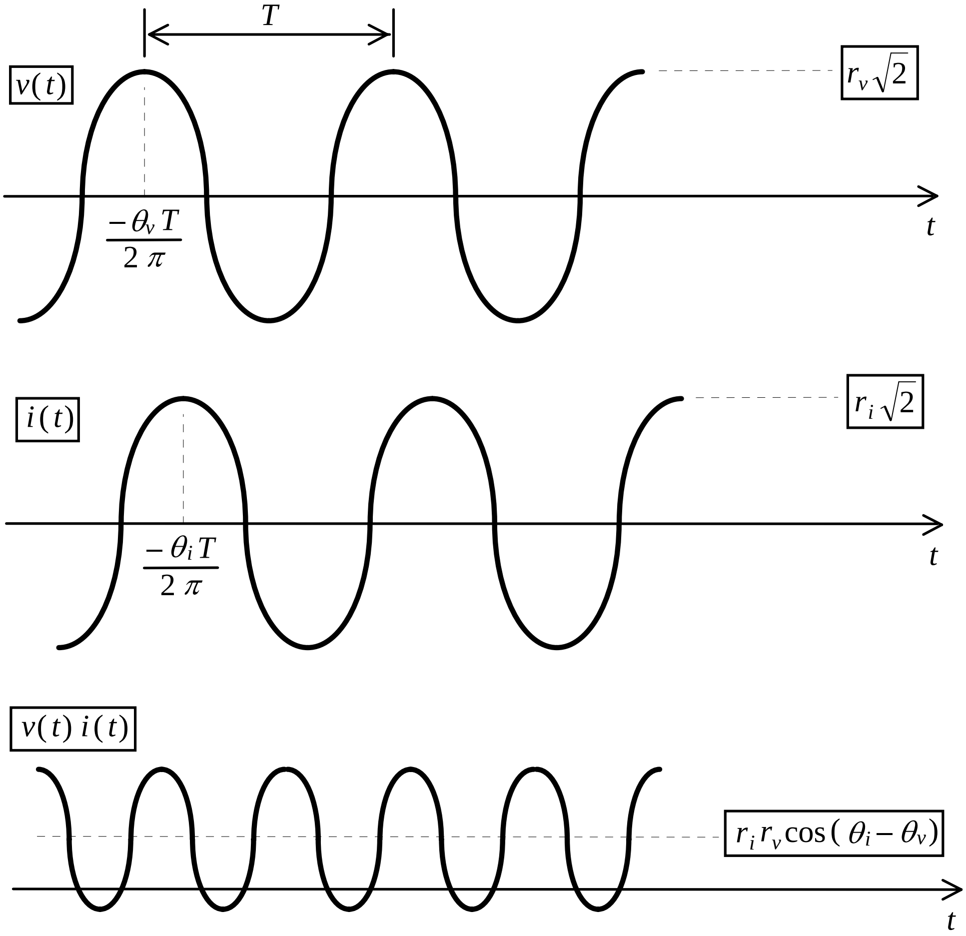
The ***average power*** consumed is

$$p_{av} = \frac{1}{1\,\text{period}}\int_{1\,\text{period}} p(t)\,dt$$

$$= \frac{r_i r_v}{T}\int_0^T \left[\cos(\theta_i - \theta_v) + \cos\left(\frac{4\pi t}{T} + \theta_i + \theta_v\right)\right]dt$$

$$= r_i r_v \cos(\theta_i - \theta_v)$$

Graphically:



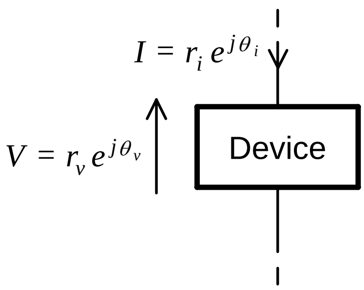
In phasor notation:

$$V = r_v e^{j\theta_v}$$

$$I = r_i e^{j\theta_i}$$

$$p_{av} = r_i r_v \cos(\theta_i - \theta_v) = \mathrm{Re}\left[r_i r_v e^{j(\theta_i - \theta_v)}\right] = \mathrm{Re}\left[r_i e^{j\theta_i} r_v e^{-j\theta_v}\right] = \mathrm{Re}\left[IV^*\right] = \mathrm{Re}\left[I^*V\right]$$

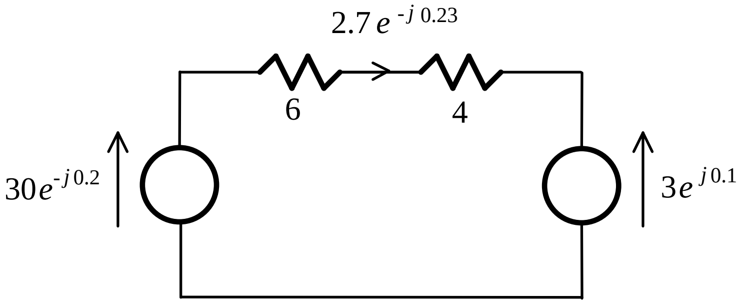
In block diagram form:



Power consumed $= \mathrm{Re}\,[V^* I] = \mathrm{Re}\,[I^* V]$
$= r_i\, r_v \cos(\theta_i - \theta_v)$

Note that the formula $p_{av} = \mathrm{Re}\left[I^*V\right]$ is based on rms voltages and currents. Also, this is valid for dc circuits, which is a special case of ac circuits with $f = 0$ and V and I having real values.

As an example, consider the ac circuit:



The power consumed are:

$$3e^{j0.1}\text{source}:\mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^*\left(3e^{j0.1}\right)\right]=\mathrm{Re}\left[8.1e^{j0.33}\right]=8.1\cos(0.33)=7.7$$

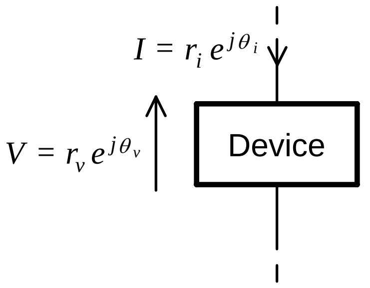
$$30e^{-j0.2}\text{source}:\mathrm{Re}\left[-\left(2.7e^{-j0.23}\right)^*\left(30e^{-j0.2}\right)\right]=-81\cos(0.03)=-81$$

$$6\ \text{resistor}:\mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^*\left(6\times 2.7e^{-j0.23}\right)\right]=6(2.7)^2=44$$

$$4\ \text{resistor}:\mathrm{Re}\left[\left(2.7e^{-j0.23}\right)^*\left(4\times 2.7e^{-j0.23}\right)\right]=4(2.7)^2=29$$

3.5 Power Factor

Consider the ac device:



Ignoring the phase difference between V and I, the voltage-current rating or ***apparent power*** consumed is

$$\text{Apparent power} = \text{voltage-current rating} = |V||I| = r_v r_i \text{ VA}$$

However, the actual power consumed is

$$\text{Actual power} = \text{Re}[V^* I] = r_v r_i \cos(\theta_i - \theta_v) \text{W}$$

The ratio of the these powers is the ***power factor*** of the device:

$$\text{Power factor} = \frac{\text{Actual power}}{\text{Apparent power}} = \cos(\theta_i - \theta_v)$$

This has a maximum value of 1 when

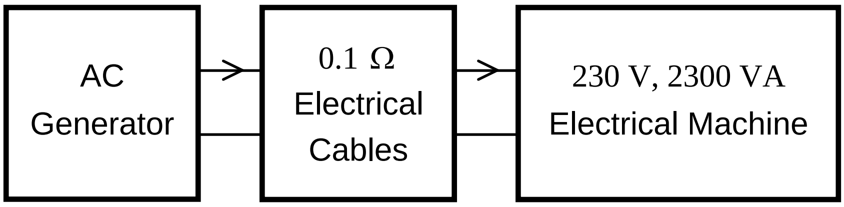
$$\text{Unity power factor} \Leftrightarrow I \text{ and } V \text{ in phase} \Leftrightarrow \theta_i = \theta_v$$

The power factor is said to be ***leading*** or ***lagging*** if

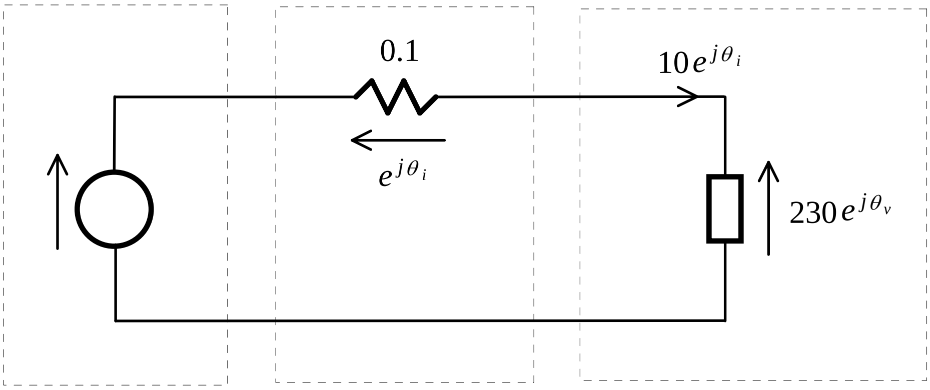
$$\text{Leading power factor} \Leftrightarrow I \text{ leads } V \text{ in phase} \Leftrightarrow \theta_i > \theta_v$$

$$\text{Lagging power factor} \Leftrightarrow I \text{ lags } V \text{ in phase} \Leftrightarrow \theta_i < \theta_v$$

Consider the following ac system:



This can be represented by:



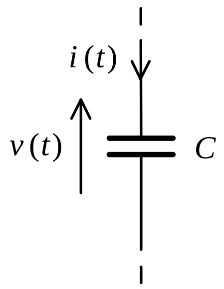
The power consumed by the machine and loss in the transmission system at different power factors are:

Voltage-current rating	$2300\,\text{VA}$	$2300\,\text{VA}$	$2300\,\text{VA}$
Voltage across machine	$230\,\text{V}$	$230\,\text{V}$	$230\,\text{V}$
Current	$10\,\text{A}$	$10\,\text{A}$	$10\,\text{A}$
Power factor	$0.11\,\text{leading}$	1	$0.11\,\text{lagging}$
$\theta_i - \theta_v$	$\cos^{-1}(0.11)$ $=1.46\text{rad}$	0	$-\cos^{-1}(0.11)$ $=-1.46\,\text{rad}$
Power consumed by machine	$(2300)(0.11)$ $= 232\,\text{W}$	$(2300)(1)$ $= 2300\,\text{W}$	$(2300)(0.11)$ $= 232\,\text{W}$
Power loss in cables	$(0.1)(10)^2$ $= 10\,\text{W}$	$(0.1)(10)^2$ $= 10\,\text{W}$	$(0.1)(10)^2$ $= 10\,\text{W}$

Clearly, the larger the power factor, the more efficient the transmission system. However, the power factor depends on the load being connected and it may be impossible for this to be specified in advance.

3.6 Capacitor

A ***capacitor*** consists of parallel plates for storing electric charges. The circuit symbol for an ideal capacitor is:



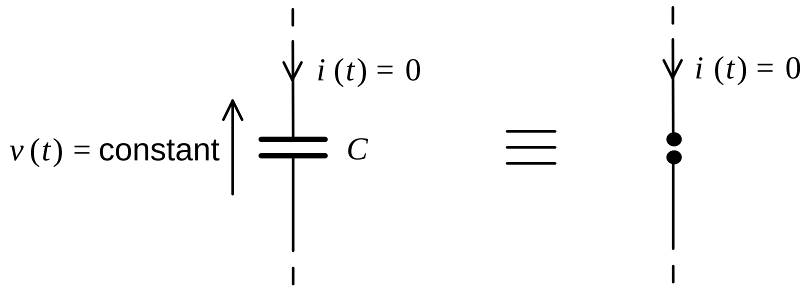
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$i(t) = C\frac{dv(t)}{dt}$$

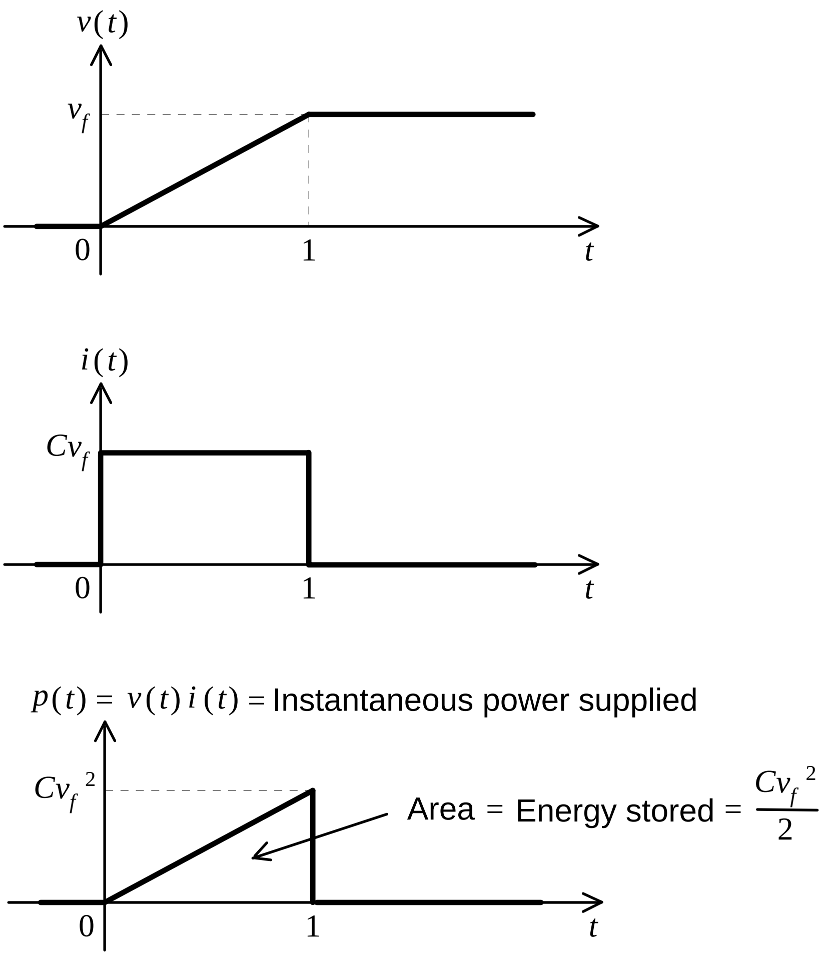
For dc circuits:

$$v(t) = \text{constant} \Rightarrow \frac{dv(t)}{dt} = 0 \Rightarrow i(t) = 0$$

and the capacitor is equivalent to an open circuit:



Consider the change in voltage, current and power supplied to the capacitor as indicated below:



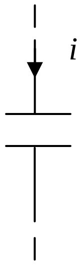
In general, the total ***energy stored*** in the electric field established by the charges on the capacitor plates at time t is

$$e(t) = \frac{Cv^2(t)}{2}$$

if the voltage is $v(t)$.

Now consider the operation of a capacitor in an ac circuit:

$$v(t) = r_v \sqrt{2} \cos(\omega t + \theta_v)$$

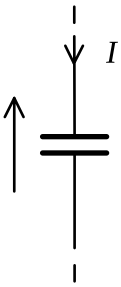


$$i(t) = C \frac{dv(t)}{dt} = -\omega C r_v \sqrt{2} \sin(\omega t + \theta_v)$$
$$= \omega C r_v \sqrt{2} \cos(\omega t + \theta_v + \frac{\pi}{2})$$

Note that since differentiating a sinusoidal function will always result in another sinusoidal function, the current $i(t)$ will be sinusoidal if the voltage $v(t)$ is sinusoidal. In phasor format:

$$I = \omega C r_v e^{j\theta_v} e^{j\frac{\pi}{2}} = j\omega C r_v e^{j\theta_v} = j\omega C V$$

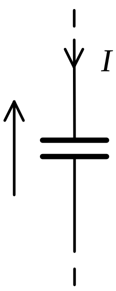
$$V = r_v e^{j\theta_v}$$



The ac voltage-current relationship for a capacitor can therefore be summarised by

$$\frac{V}{I} = \frac{1}{j\omega C}$$

or symbolically:



With phasor representation, the capacitor behaves as if it is a resistor with a "complex resistance" or an ***impedance*** of

$$Z_C = \frac{1}{j\omega C}$$

This impedance is purely imaginary and the average power absorbed is

$$p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* I Z_C\right] = \mathrm{Re}\left[\frac{|I|^2}{j\omega C}\right] = 0$$

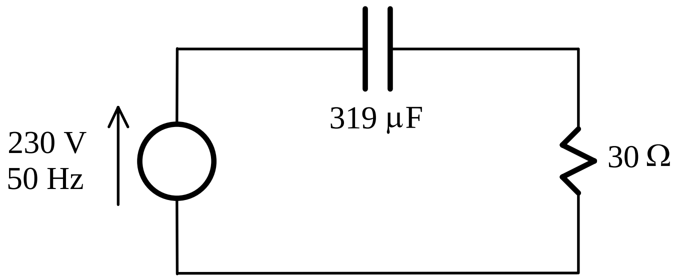
which confirms that an ideal capacitor is a non-dissipative but energy-storing device.

Since the phase of I relative to that of V is

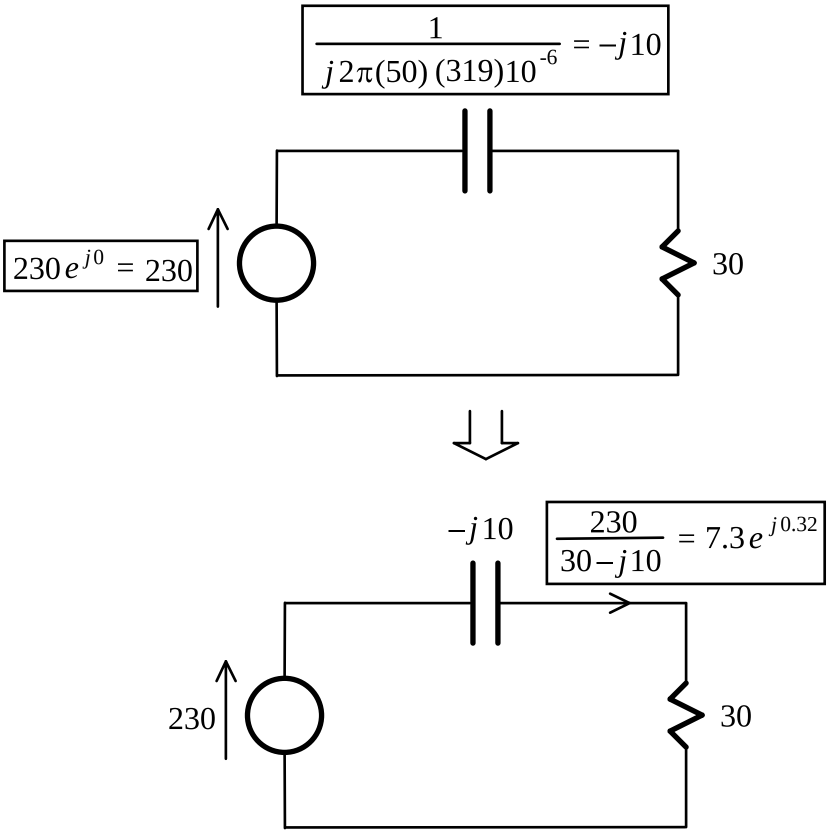
$$\mathrm{Arg}[I] - \mathrm{Arg}[V] = \mathrm{Arg}\left[\frac{I}{V}\right] = \mathrm{Arg}\left[\frac{1}{Z_C}\right] = \mathrm{Arg}[j\omega C] = 90^0$$

the ac current $i(t)$ across the capacitor leads the voltage $v(t)$ by 90^0.

As an example, consider the following ac circuit:



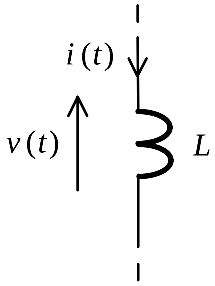
In phasor notation (taking the source to have a reference phase of 0):



Total circuit impedance	$Z = (30 - j10)$
Total circuit ***reactance***	$X = \mathrm{Im}[Z] = \mathrm{Im}[30 - j10] = -10\,\Omega$
Total circuit resistance	$R = \mathrm{Re}[Z] = \mathrm{Re}[30 - j10] = 30\,\Omega$
Current (rms)	$\lvert I \rvert = 7.3\,\mathrm{A}$
Current (peak)	$\sqrt{2}\lvert I \rvert = 7.3\sqrt{2} = 10\,\mathrm{A}$
Source *V*-*I* phase relationship	I leads by 0.32 rad
Power factor of entire circuit	$\cos(0.32) = 0.95$ leading
Power supplied by source	$\mathrm{Re}\left[(230)^*\left(7.3e^{j0.32}\right)\right] = (230)(7.3)\cos(0.32) = 1.6\mathrm{kW}$
Power consumed by resistor	$\mathrm{Re}\left[\left(7.3e^{j0.32}\right)^*\left(30 \times 7.3e^{j0.32}\right)\right] = (7.3)^2 30 = 1.6\mathrm{kW}$

3.7 Inductor

An ***inductor*** consists of a coil of wires for establishing a magnetic field. The circuit symbol for an ideal inductor is:



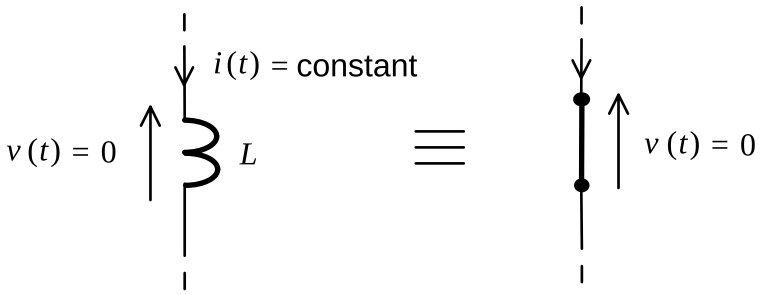
Provided that the voltage and current arrows are in opposite directions, the voltage-current relationship is:

$$v(t) = L\frac{di(t)}{dt}$$

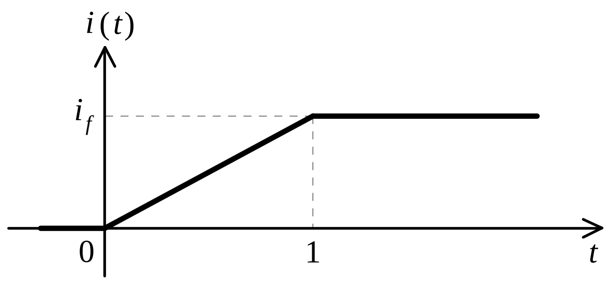
For dc circuits:

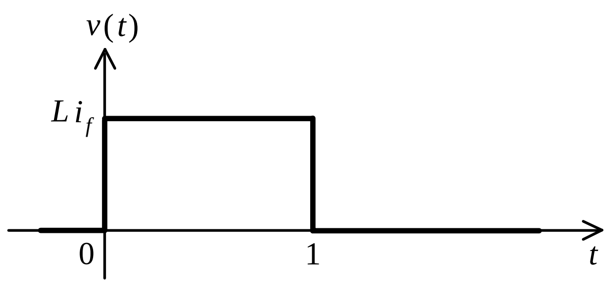
$$i(t) = \text{constant} \Rightarrow \frac{di(t)}{dt} = 0 \Rightarrow v(t) = 0$$

and the inductor is equivalent to a short circuit:

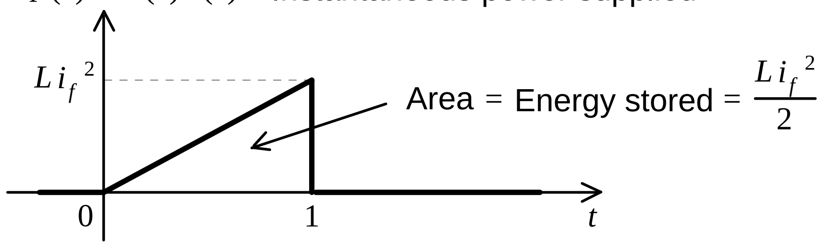


Consider the change in voltage, current and power supplied to the inductor as indicated below:





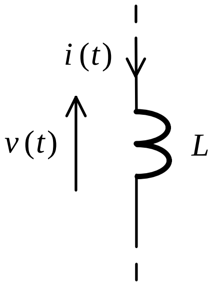
$p(t) = v(t)\, i(t) =$ Instantaneous power supplied



In general, the total energy stored in the magnetic field established by the current $i(t)$ in the inductor at time t is given by

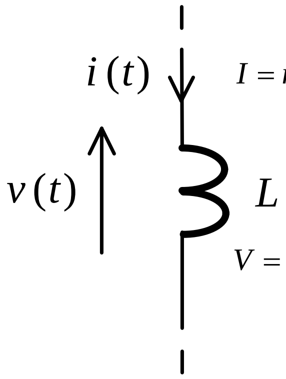
$$e(t) = \frac{Li^2(t)}{2}$$

Now consider the operation of an inductor in an ac circuit:



$$i(t) = r_i\sqrt{2}\cos(\omega t + \theta_i)$$

$$v(t) = L\frac{di(t)}{dt} = -\omega L r_i\sqrt{2}\sin(\omega t + \theta_i)$$
$$= \omega L r_i\sqrt{2}\cos(\omega t + \theta_i + \frac{\pi}{2})$$



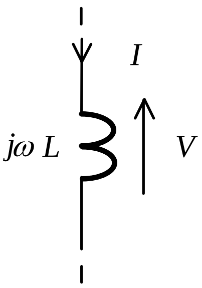
$$I = r_i e^{j\theta_i}$$

$$V = \omega L r_i e^{j\theta_i} e^{j\pi/2} = j\omega L r_i e^{j\theta_i} = (j\omega L)I$$

The ac voltage-current relationship for an inductor can therefore be summarised by

$$\frac{V}{I} = j\omega L$$

or symbolically:



Thus, in ac circuits, the inductor has an impedance of

$$Z_L = j\omega L$$

and will absorb an average power of

$$p_{av} = \mathrm{Re}\left[I^* V\right] = \mathrm{Re}\left[I^* Z_L I\right] = \mathrm{Re}\left[j\omega L I^* I\right] = \mathrm{Re}\left[j\omega L|I|^2\right] = 0$$

which shows that an ideal inductor is a non-dissipative but energy-storing device.

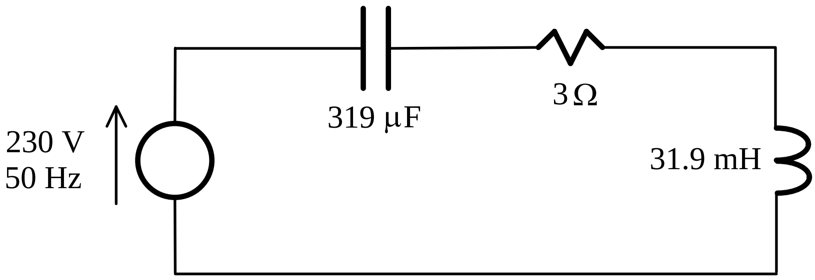
Since the phase of I relative to that of V is

$$\text{Arg}[I]-\text{Arg}[V]=\text{Arg}\left[\frac{I}{V}\right]=\text{Arg}\left[\frac{1}{Z_L}\right]=\text{Arg}\left[\frac{1}{j\omega L}\right]=-90^0$$

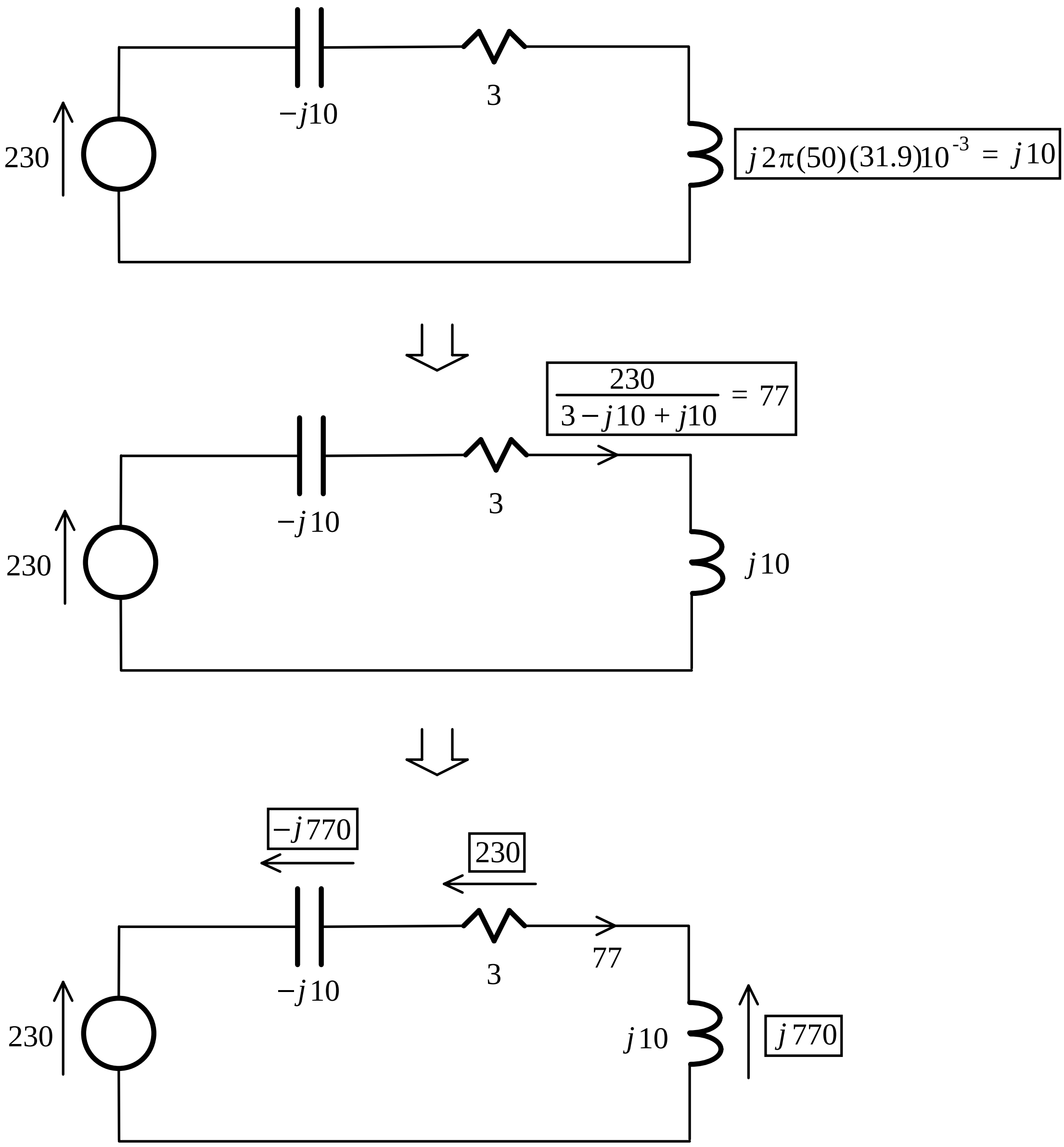
the ac current $i(t)$ lags the voltage $v(t)$ by 90^0.

Since most motors consist of coils of windings, most ac loads are inductive in nature.

As an example, consider the following series ac circuit:



Using phasors:



Total circuit impedance	$Z = 3 - j10 + j10 = 3\Omega$
Total circuit reactance	$X = \mathrm{Im}[Z] = \mathrm{Im}[3] = 0\Omega$
Total circuit resistance	$R = \mathrm{Re}[Z] = \mathrm{Re}[3] = 3\Omega$
Current (rms)	$\lvert I \rvert = 77\,\mathrm{A}$
Current (peak)	$\sqrt{2}\lvert I \rvert = 77\sqrt{2} = 108\,\mathrm{A}$
Source voltage-current phase relationship	$0(\text{in phase})$
Power factor of entire circuit	$\cos(0) = 1$
Power supplied by source	$\mathrm{Re}[(77)^*(230)] = 18\mathrm{kW}$
Power consumed by resistor	$\mathrm{Re}[(77)^*(3 \times 77)] = 18\mathrm{kW}$

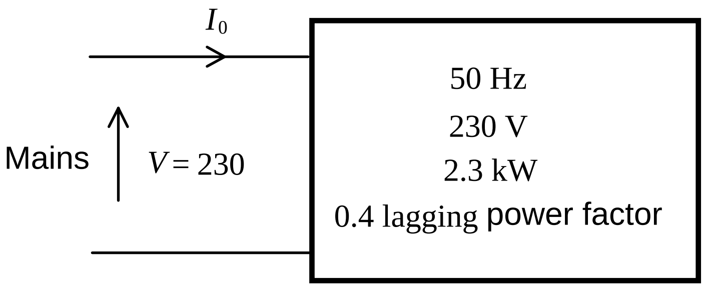
Note that the rms voltages across the inductor and capacitor are larger than the source voltage. This is possible in ac circuits because the reactances of capacitors and inductors, and so the voltages developed across them, may cancel out one another:

Source voltage		Voltage across capacitor		Voltage across resistor		Voltage across inductor
230	$=$	$-j770$	$+$	230	$+$	$j770$

In dc circuits, it is not possible for a passive resistor (with positive resistance) to cancel out the effect of another passive resistor (with positive resistance).

3.8 Power Factor Improvement

Consider the following system:



Electrical Machine

The current I_0 can be found as follows:

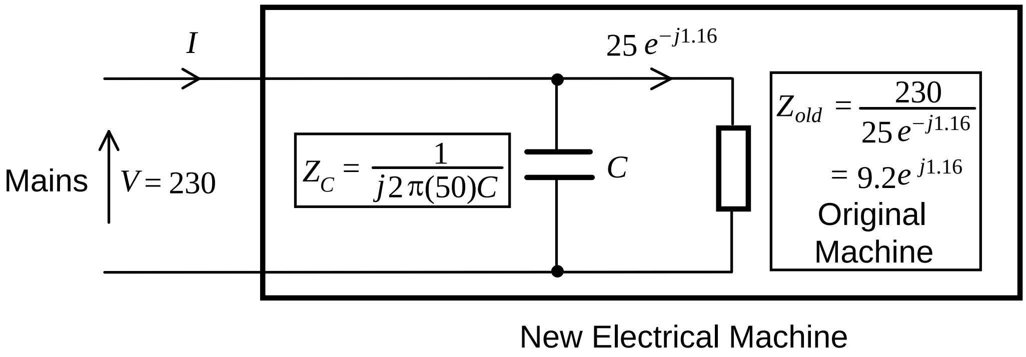
$$\frac{2300\text{W}}{(230\text{V})(|I_0|\text{A})} = 0.4 \Rightarrow |I_0| = \frac{2300}{(230)(0.4)} = 25$$

$$\left.\begin{array}{l}\cos\{\text{Arg}[I_0] - \text{Arg}[V]\} = 0.4 \\ \text{Arg}[I_0] - \text{Arg}[V] < 0\end{array}\right\} \Rightarrow \text{Arg}[I_0] = -\cos^{-1}(0.4) = -1.16$$

$$I_0 = |I_0| e^{j\text{Arg}[I_0]} = 25e^{-j1.16}$$

Due to the small power factor, the machine cannot be connected to standard 13A outlets even though it consumes only 2.3kW of power.

To overcome this problem, a parallel capacitor can be used to improve the power factor:



$$I = \frac{V}{Z_C} + 25e^{-j1.16} = j23000\pi C + 10 - j23 = 10 + j(23000\pi C - 23)$$

Thus, if we choose

$$23000\pi C = 23 \;\Rightarrow\; C = 0.32\,\text{mF}$$

then

$$I = 10\,\text{A}$$

Power factor of new machine $= \cos[\text{Arg}(I) - \text{Arg}(V)] = 1$

By changing the power factor, the improved machine can now be connected to standard 13A outlets. The price to pay is the use of an additional capacitor.

To reduce cost, we may wish to use a capacitor which is as small as possible. To find the smallest capacitor that will satisfy the 13A requirement:

$$|I|^2 = 10^2 + (72200C - 23)^2 = 13^2$$

$$13^2 = 10^2 + (72200C - 23)^2$$

$$0 = 10^2 - 13^2 + (72200C - 23)^2 = (72200C - 23)^2 - 8.3^2$$

$$0 = (72200C - 23 - 8.3)(72200C - 23 + 8.3)$$

$C = 0.2\,\text{mF}$ or $0.44\,\text{mF}$

There are 2 possible values for C, one giving a lagging overall power factor, the other giving a leading overall power factor. To save cost, C should be

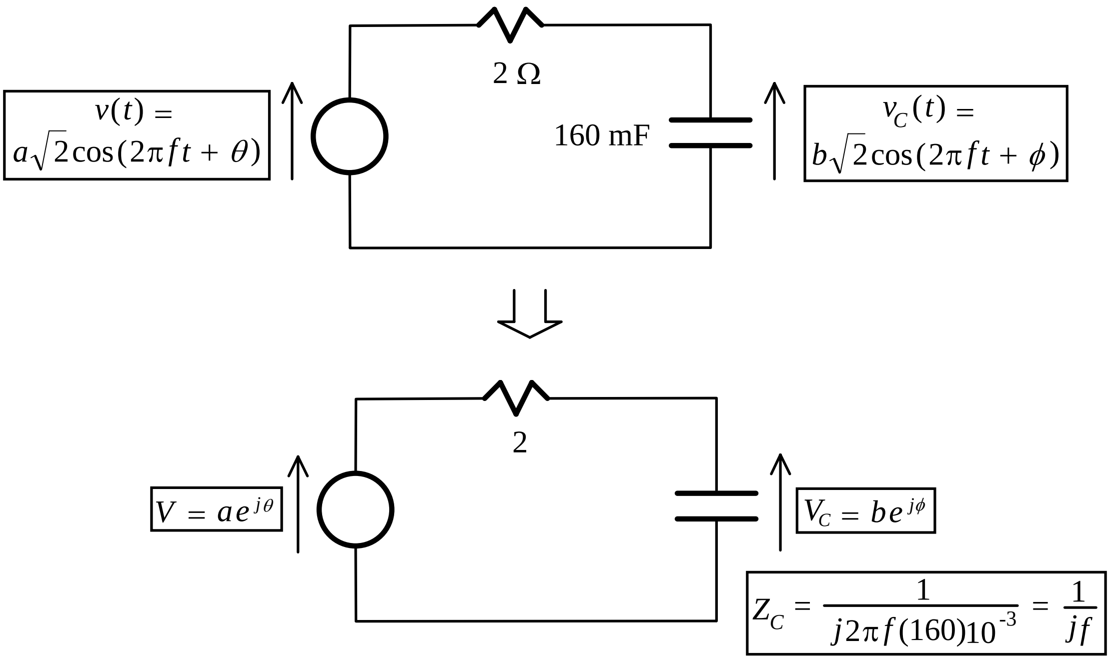
$$C = 0.2\,\text{mF}$$

4

FREQUENCY RESPONSE

4.1 RC Circuit

Consider the ***series RC circuit***:



Taking the input/excitation to be a sinusoid at frequency f and represented by the phasor V, the output/response will be sinusoidal and can be represented by the phasor V_C. The ratio of the output phasor to the input phasor or voltage gain of the circuit is

$$H(f) = \frac{V_C}{V} = \frac{be^{j\phi}}{ae^{j\theta}} = \frac{b}{a} e^{j(\phi-\theta)} = \frac{Z_C}{2+Z_C} = \frac{\frac{1}{jf}}{2+\frac{1}{jf}} = \frac{1}{1+j2f}$$

This is a function of frequency and is called the ***frequency response*** of the circuit.

The magnitude of $H(f)$ is

$$|H(f)| = \left|\frac{V_C}{V}\right| = \frac{|V_C|}{|V|} = \frac{b}{a}$$

$$= \sqrt{\frac{1}{1+(2f)^2}} = \sqrt{\frac{1}{1+4f^2}}$$

and is called the ***magnitude response***.

The phase of $H(f)$ is

$$\mathrm{Arg}[H(f)] = \mathrm{Arg}\left[\frac{V_C}{V}\right] = \mathrm{Arg}[V_C] - \mathrm{Arg}[V] = \phi - \theta$$

$$= \mathrm{Arg}\left[\frac{1}{1+j2f}\right] = -\mathrm{Arg}[1+j2f] = -\tan^{-1}(2f)$$

and is called the ***phase response***.

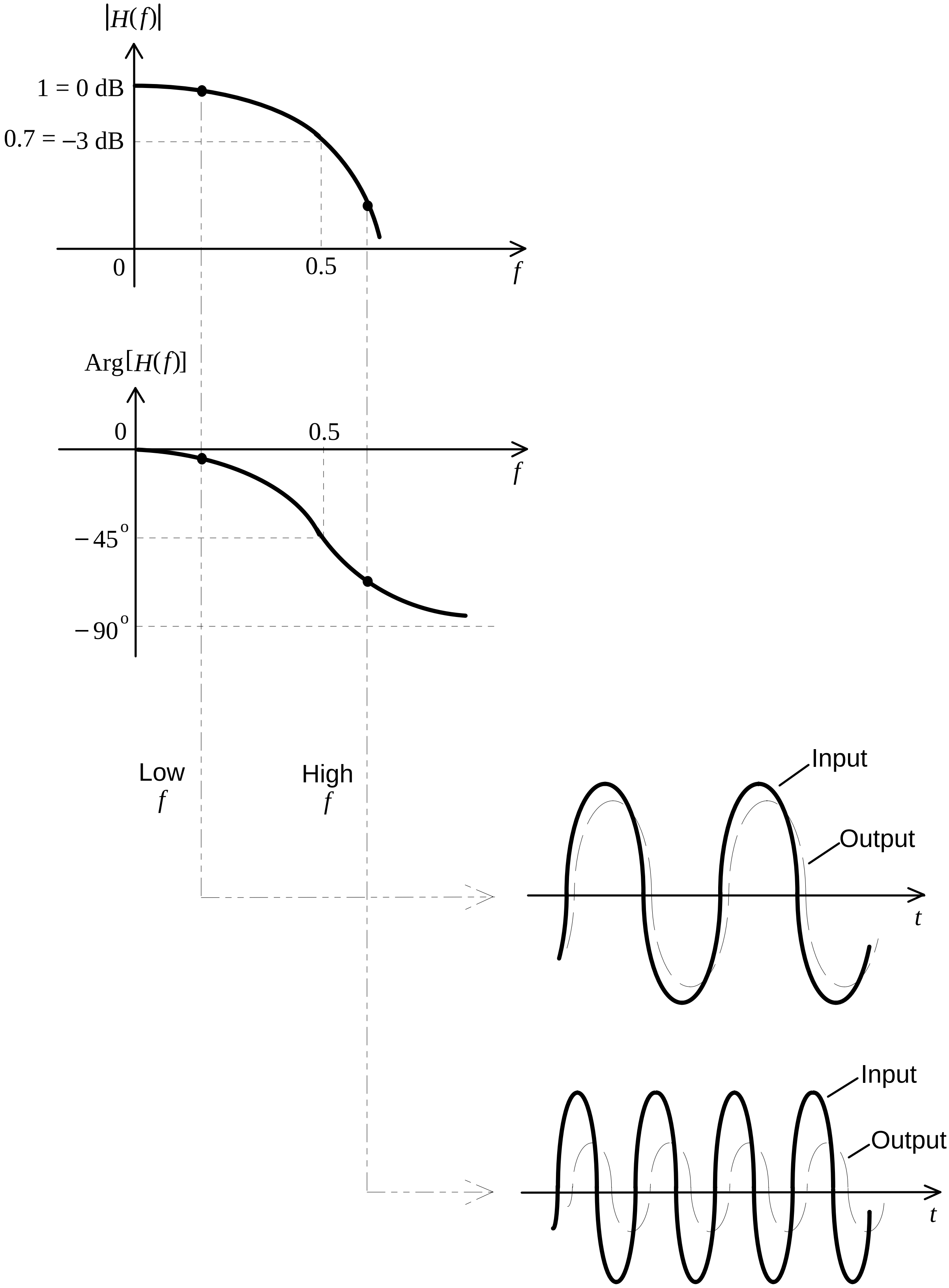
The physical significance of these responses is that $H(f)$ gives the ratio of output to input phasors, $|H(f)|$ gives the ratio of output to input magnitudes, and $\mathrm{Arg}[H(f)]$ gives the output to input phase difference at a frequency f. Thus, for the RC circuit:

Input	$v(t)=3\sqrt{2}\cos[2\pi(5)t+7]$ $V=3e^{j7}$	$v(t)=r\sin[2\pi(4)t]=r\cos\left[2\pi(4)t-\frac{\pi}{2}\right]$ $V=\frac{r}{\sqrt{2}}e^{-j\pi/2}$
Frequency	$f=5$	$f=4$
Frequency response	$H(5)=\frac{1}{1+j10}$	$H(4)=\frac{1}{1+j8}$
Magnitude response	$\|H(5)\|=\frac{1}{\sqrt{101}}$	$\|H(4)\|=\frac{1}{\sqrt{65}}$
Phase response	$\text{Arg}[H(5)]=-\tan^{-1}(10)$	$\text{Arg}[H(4)]=-\tan^{-1}(8)$
Output	$v_C(t)=\frac{3\sqrt{2}}{\sqrt{101}}\cos[2\pi(5)t+7-\tan^{-1}(10)]$ $V_C=\frac{3}{\sqrt{101}}e^{j[7-\tan^{-1}(10)]}$	$v_C(t)=\frac{r}{\sqrt{65}}\sin[2\pi(4)t-\tan^{-1}(8)]$ $=\frac{r}{\sqrt{65}}\cos\left[2\pi(4)t-\tan^{-1}(8)-\frac{\pi}{2}\right]$ $V_C=\frac{r}{\sqrt{130}}e^{j[-\tan^{-1}(8)-\pi/2]}$

Due to the presence of components such as capacitors and inductors with frequency-dependent impedances, $H(f)$ is usually frequency-dependent and the characteristics of the circuit is often studied by finding how $H(f)$ changes as f is varied. Numerically, for the series RC circuit:

f	$\|H(f)\|=\frac{1}{\sqrt{(1+4f^2)}}$	$\text{Arg}[H(f)]=-\tan^{-1}(2f)$
0	$1=20\log(1)=0\text{dB}$	$0\,\text{rad}=0^0$
0.5	$\frac{1}{\sqrt{1+4(0.5)^2}}=\frac{1}{\sqrt{2}}=20\log\left(\frac{1}{\sqrt{2}}\right)=-3\text{dB}$	$-\tan^{-1}(2\times0.5)=-\frac{\pi}{4}\text{rad}=-45^0$
$\to\infty$	$\to 0=-\infty\,\text{dB}$	$-\tan^{-1}(\infty)=-\frac{\pi}{2}\text{rad}=-90^0$

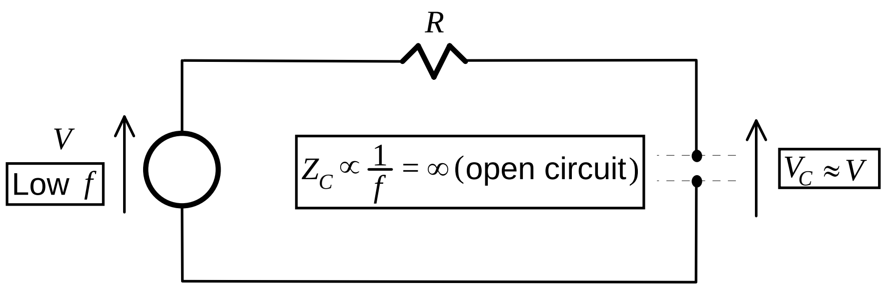
Graphically:



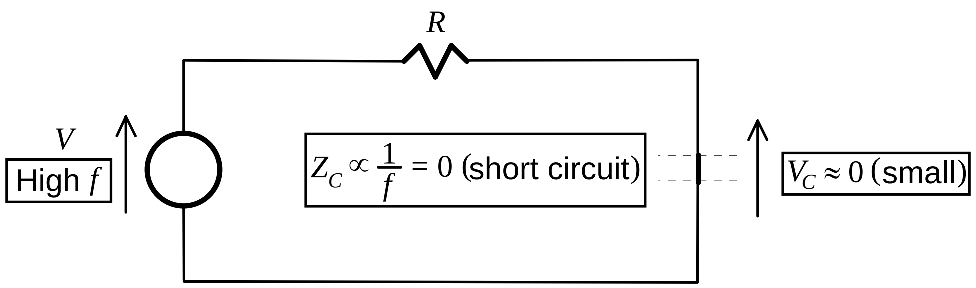
At small f, the output approximates the input. However, at high f, the output will become attenuated. Thus, the circuit has a ***lowpass*** characteristic (low frequency input will be passed, high frequency input will be rejected).

The frequency at which $|H(f)|$ falls to -3dB of its maximum value is called the ***cutoff*** frequency. For the above example, the cutoff frequency is 0.5Hz.

To see why the circuit has a lowpass characteristic, note that at low f, C has large impedance (approximates an open circuit) when compare with R (2 in the above example). Thus, V_C will be approximately equal to V:

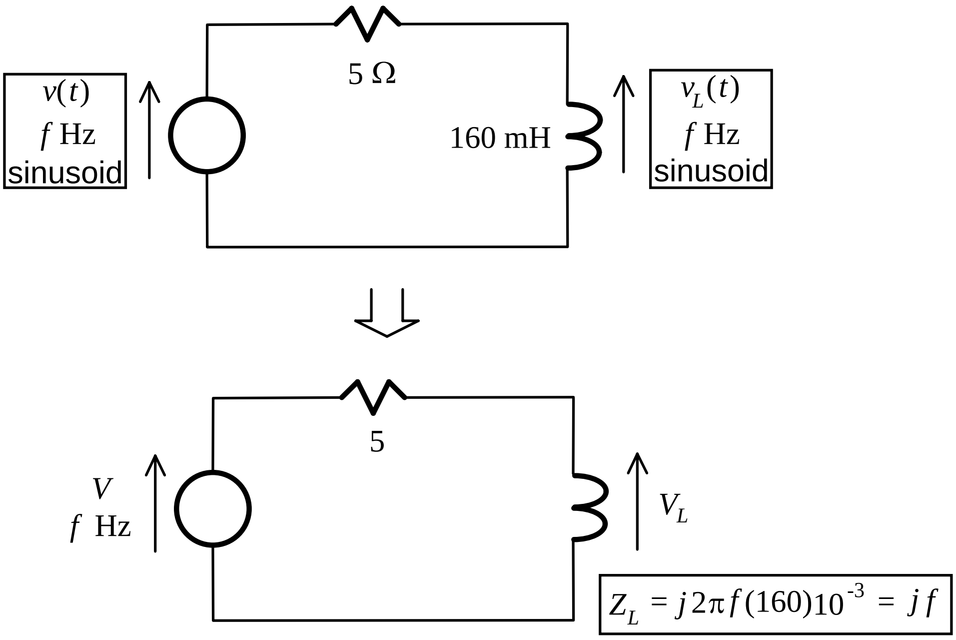


However, at high f, C has small impedance (approximates a short circuit) when compare with R. Thus, V_C will be small:



4.2 RL Circuit

Consider the ***series RL circuit***:



With V being the input and V_L being the output, the frequency response is

$$\frac{V_L}{V}=H(f)=\frac{Z_L}{5+Z_L}=\frac{jf}{5+jf}$$

The magnitude response is

$$|H(f)|=\sqrt{\frac{f^2}{5^2+f^2}}=\sqrt{\frac{f^2}{25+f^2}}$$

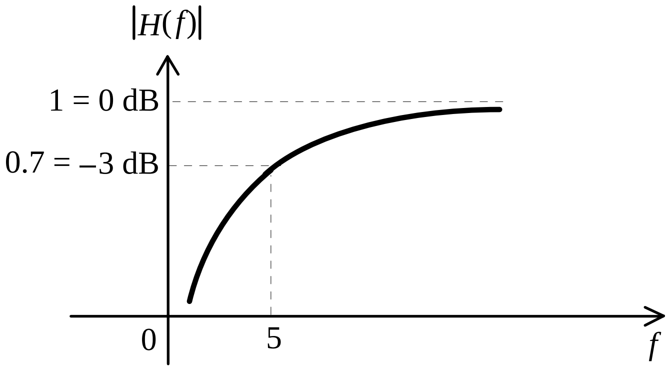
The phase response is

$$\text{Arg}[H(f)]=\text{Arg}\left[\frac{jf}{5+jf}\right]=\text{Arg}[jf]-\text{Arg}[5+jf]=\frac{\pi}{2}-\tan^{-1}\left(\frac{f}{5}\right)$$

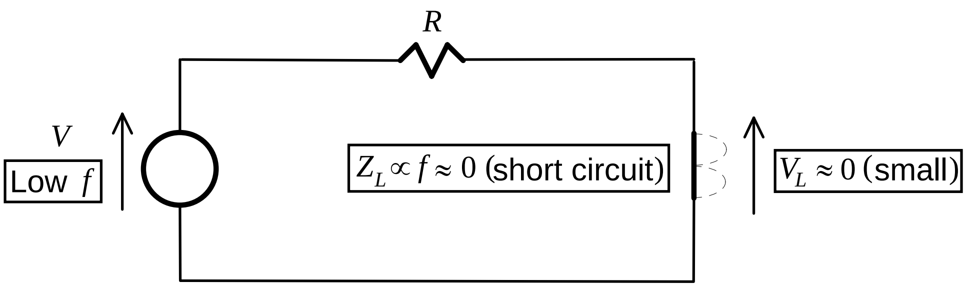
Numerically:

f	$\lvert H(f)\rvert = \sqrt{\dfrac{f^2}{25+f^2}}$	$\text{Arg}[H(f)] = \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{f}{5}\right)$
0	$0 = 20\log(0) = -\infty\text{dB}$	$\dfrac{\pi}{2}\text{rad} = 90^0$
5	$\sqrt{\dfrac{5^2}{25+5^2}} = \dfrac{1}{\sqrt{2}} = 20\log\left(\dfrac{1}{\sqrt{2}}\right) = -3\text{dB}$	$\dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{5}{5}\right) = \dfrac{\pi}{4}\text{rad} = 45^0$
$\to\infty$	$\to 1 = 0\text{dB}$	$\dfrac{\pi}{2} - \tan^{-1}(\infty) = 0\text{rad} = 0^0$

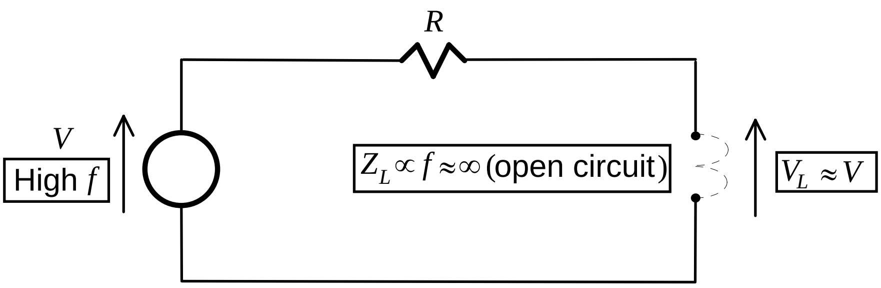
Graphically:



Physically, at small f, L has small impedance (approximates a short circuit) when compare with R (5 in the above example). Thus, V_L will be small:



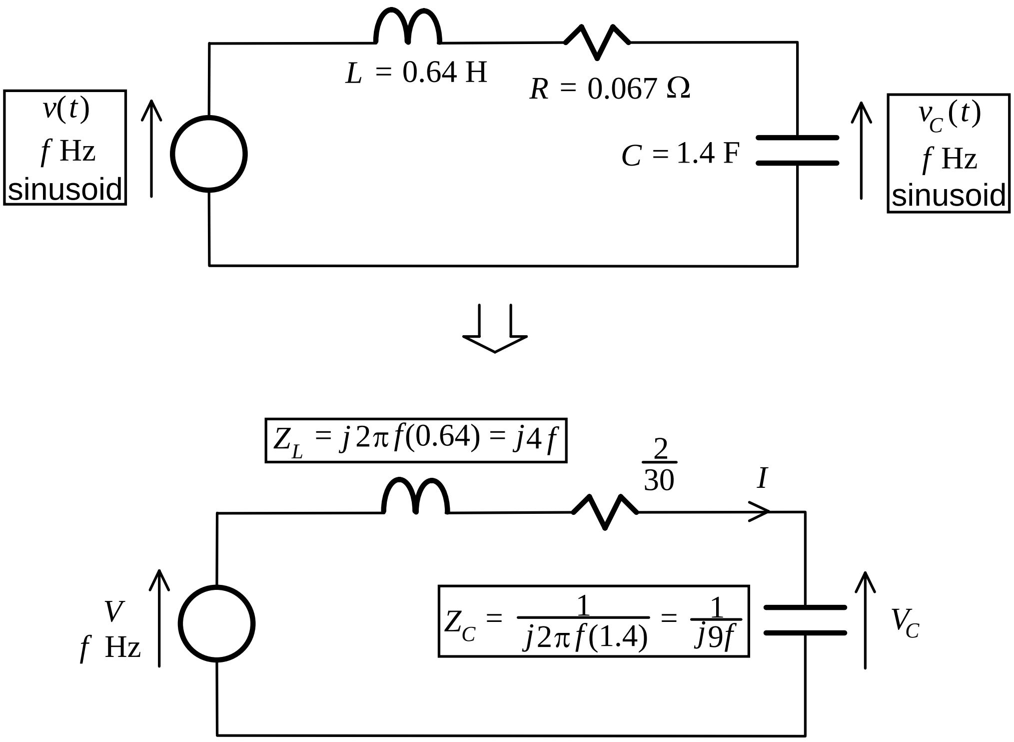
However, at high f, L has large impedance (approximates an open circuit) when compare with R. Thus, V_L will approximates V:



Due to these characteristics, the circuit is ***highpass*** in nature.

4.3 Tune Circuit

Consider the ***series tune circuit***:



The total circuit impedance is

$$Z = R + Z_L + Z_C = \frac{2}{30} + j4f + \frac{1}{j9f} = \frac{2}{30} + j\left(4f - \frac{1}{9f}\right)$$

At a frequency $f = f_0$ given by

$$f_0 = \frac{1}{\sqrt{(4)(9)}} = \frac{1}{6} \Leftrightarrow f_0 = \frac{1}{\sqrt{(2\pi L)(2\pi C)}} = \frac{1}{2\pi\sqrt{LC}}$$

$Z_L = j4/6 = j2/3$ cancels with $Z_C = 1/(j9/6) = -j2/3$, $Z = 2/30$ becomes purely resistive, and the circuit is said to be in ***resonance***.

The ratio

$$Q = \frac{\text{Reactance of inductor at } f_0}{\text{Resistance}} = \frac{2/3}{2/30} = 10 \Leftrightarrow Q = \frac{2\pi f_0 L}{R}$$

is called the ***Q factor***.

With V being the input and V_C being the output, the frequency response is

$$H(f)=\frac{V_C}{V}=\frac{Z_C}{Z}=\frac{\frac{1}{j9f}}{\frac{2}{30}+j4f+\frac{1}{j9f}}=\frac{1}{1-36f^2+j0.6f}$$

The magnitude response is

$$|H(f)|=\frac{1}{\sqrt{(1-36f^2)^2+(0.6f)^2}}=\frac{1}{\sqrt{(36f^2)^2-(72-0.6^2)f^2+1}}$$

$$=\frac{1}{\sqrt{(36f^2)^2-2(36f^2)\left(1-\frac{0.6^2}{72}\right)+\left(1-\frac{0.6^2}{72}\right)^2+1-\left(1-\frac{0.6^2}{72}\right)^2}}$$

$$=\frac{1}{\sqrt{\left[36f^2-\left(1-\frac{0.6^2}{72}\right)\right]^2+\left(\frac{0.6^2}{72}\right)\left(2-\frac{0.6^2}{72}\right)}}$$

Since f only appears in the $[\bullet]^2$ term in the denominator and $[\bullet]^2\geq 0$, $|H(f)|$ will increase if $[\bullet]^2$ becomes smaller, and vice versa. The maximum value for $|H(f)|$ corresponds to the situation of $[\bullet]^2=0$ or at a frequency $f=f_{peak}$ given by:

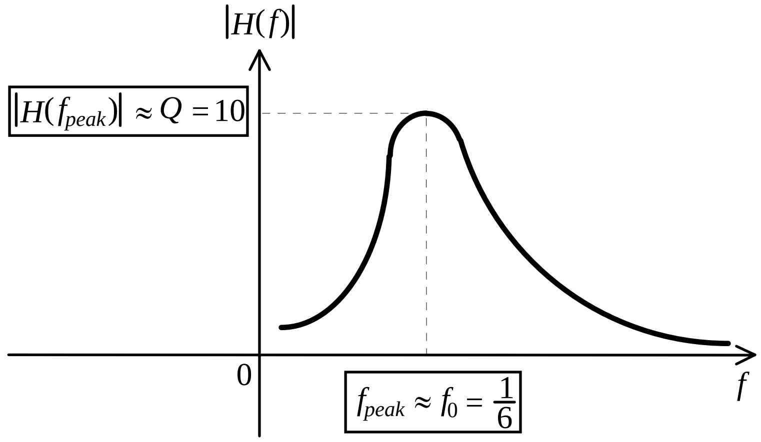
$$36f_{peak}{}^2=1-\frac{0.6^2}{72}\approx 1\Leftrightarrow f_{peak}\approx\frac{1}{6}\Leftrightarrow f_{peak}\approx f_0$$

When f moves away from f_{peak}, the value of $[\bullet]^2$ will increase and $|H(f)|$ will decrease.

At $f=f_{peak}$, $[\bullet]^2=0$ and the maximum value for $|H(f)|$ is

$$|H(f_{peak})|=\frac{1}{\sqrt{\left(\frac{0.6^2}{72}\right)\left(2-\frac{0.6^2}{72}\right)}}\approx\frac{1}{\sqrt{\left(\frac{0.6^2}{72}\right)(2)}}=10\Leftrightarrow|H(f_{peak})|\approx Q$$

Graphically:

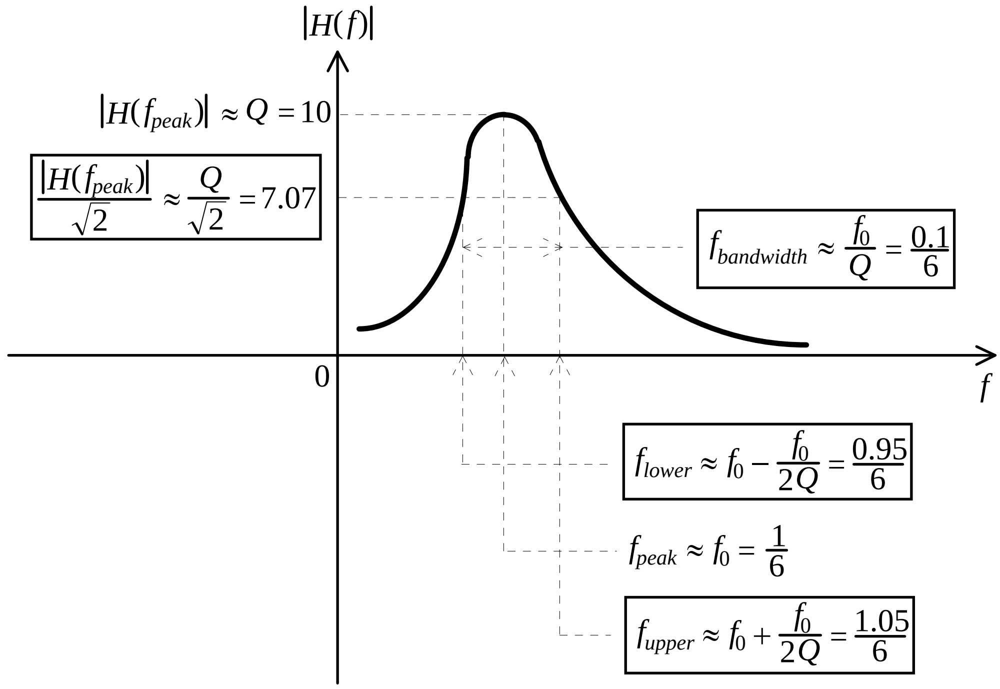


The series tuned circuit has a ***bandpass*** characteristic. Low- and high-frequency inputs will get attenuated, while inputs close to the resonant frequency will get amplified by a factor of approximately Q.

The cutoff frequencies, at which $|H(f)|$ decrease by a factor of $\sqrt{2}$ or by 3dB from its peak value $|H(f_{peak})|$, can be shown to be given by

$$f_{lower} \approx f_0\left(1 - \frac{1}{2Q}\right) \quad \text{and} \quad f_{upper} \approx f_0\left(1 + \frac{1}{2Q}\right)$$

Graphically:



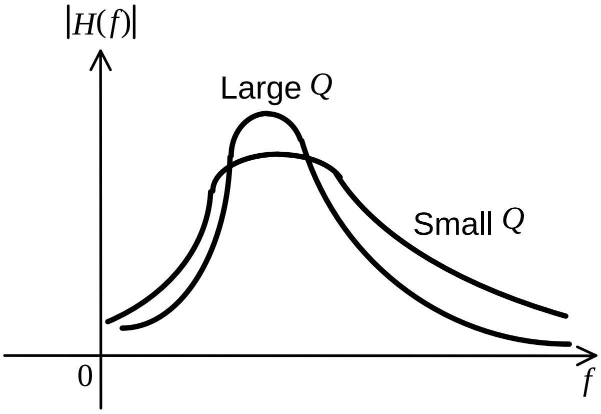
Very roughly, the circuit will pass inputs with frequencies between f_{lower} and f_{upper}. The ***bandwidth*** of the circuit is

$$f_{bandwidth} = f_{upper} - f_{lower} \approx \frac{f_0}{Q}$$

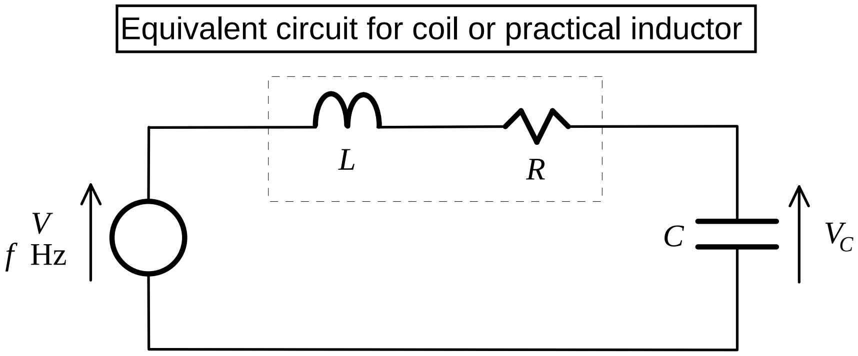
and the ***fractional bandwidth*** is

$$\frac{f_{bandwidth}}{f_0} \approx \frac{1}{Q}$$

The larger the Q factor, the sharper the magnitude response, the bigger the amplification, and the narrower the fractional bandwidth:



In practice, a series tune circuit usually consists of a practical inductor or coil connected in series with a practical capacitor. Since a practical capacitor usually behaves quite closely to an ideal one but a coil will have winding resistance, such a circuit can be represented by:



The main features are:

Circuit impedance	$Z = R + j2\pi fL + \frac{1}{j2\pi fC}$
Resonance frequency	$f_0 = \frac{1}{2\pi\sqrt{LC}}$
Q factor	$Q = \frac{2\pi f_0 L}{R}$
Frequency response	$H(f) = \frac{1}{1-4\pi^2 f^2 LC + j2\pi fCR} = \frac{1}{1-\left(\frac{f}{f_0}\right)^2 + \frac{j}{Q}\left(\frac{f}{f_0}\right)}$

For the usual situation when Q is large:

Magnitude response	Bandpass with $\lvert H(f)\rvert$ decreasing as $f \to 0$ and $f \to \infty$
Response peak	$\lvert H(f)\rvert$ peaks at $f = f_{peak} \approx f_0$ with $\lvert H(f_{peak})\rvert \approx Q$
Cutoff frequencies	$\lvert H(f)\rvert = \frac{\lvert H(f_{peak})\rvert}{\sqrt{2}} \approx \frac{Q}{\sqrt{2}}$ at $f = f_{lower}, f_{upper} \approx f_0\left(1-\frac{1}{2Q}\right), f_0\left(1+\frac{1}{2Q}\right)$
Bandwidth	$f_{bandwidth} = f_{upper} - f_{lower} \approx \frac{f_0}{Q}$
Fractional bandwidth	$\frac{f_{bandwidth}}{f_0} \approx \frac{1}{Q}$

The Q factor is an important parameter of the circuit. As defined above:

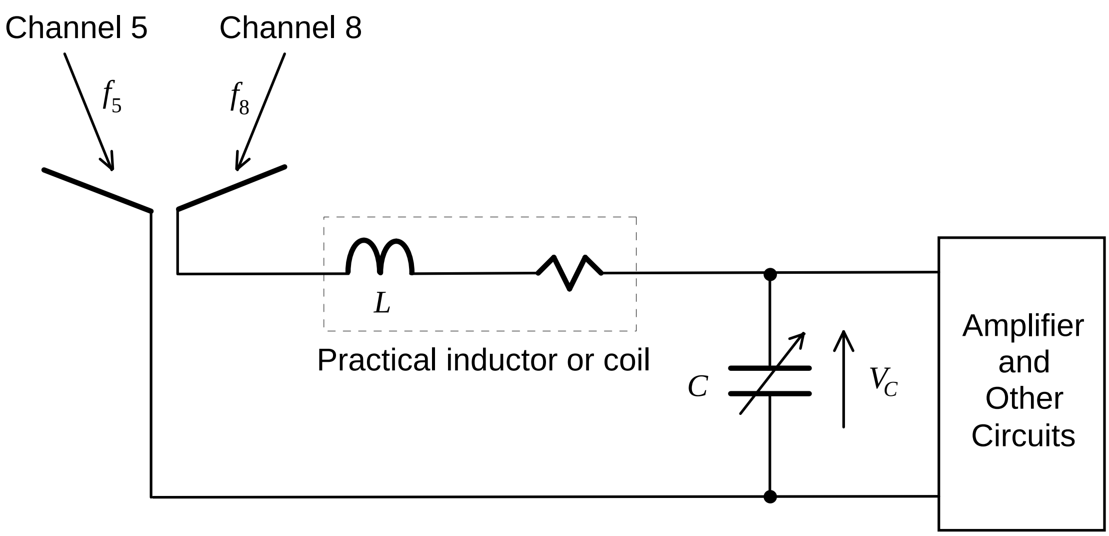
$$Q = \frac{2\pi f_0 L}{R} = \frac{\text{Inductor reactance at } f_0}{\text{Circuit resistance}}$$

However, since R is usually the winding resistance of the practical coil making up the tune circuit:

$$Q = \frac{\text{Reactance of practical coil at } f_0}{\text{Resistance of practical coil}}$$

As a good practical coil should have low winding resistance and high inductance, the Q factor is often taken to be a characteristic of the practical inductor or coil. The higher the Q factor, the higher the ***quality*** of the coil.

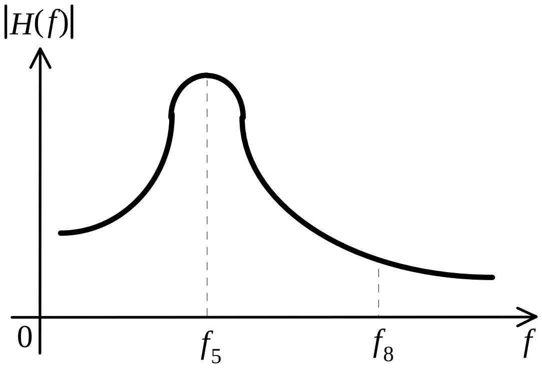
Due to its bandpass characteristic, tune circuits are used in radio and tv tuners for selecting the frequency channel of interest:



To tune in to channel 5, C has to be adjusted to a value of C_5 so that the circuit resonates at a frequency given by

$$f_5 = \frac{1}{2\pi\sqrt{LC_5}}$$

and has a magnitude response of:



To tune in to channel 8, C has to be adjusted to a value of C_8 so that the circuit resonates at a frequency given by

$$f_8 = \frac{1}{2\pi\sqrt{LC_8}}$$

and has a magnitude response of: