Decompositions of Proper Systems
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This review is from: Linear Systems Theory: A Structural Decomposition Approach (Control Engineering) (Hardcover)

This book is part of an advanced and innovative presentation of certain topics in system theory and control system design that are very rarely addressed elsewhere (I actually do not know of another source that addresses some of these issues). One might use this book to ask questions like "How do I select actuators/sensors so that transmission zeros are favorable to control design?". Those who have reached a level of knowledge and maturity in systems and control know that this is vital.

This book is not an introductory text into linear system theory. One might read this text alongside other "introductory" linear system theory books by C.T. Chen, Antsaklis and Michel, or Callier and Desoer, etc. This book is mathematically rigorous and there are many special definitions and symbols. The author makes an effort to extend the presentation to engineers with computer codes and numerical examples, which are helpful. Nonetheless, I still feel as though my head is about to explode when I read this book. It is difficult.
Theorem 5.2.1. Consider the SISO system of (5.2.1). There exist nonsingular state, input and output transformations $\Gamma_s \in \mathbb{R}^{n \times n}$, $\Gamma_i \in \mathbb{R}$ and $\Gamma_o \in \mathbb{R}$, which decompose the state space of $\Sigma$ into two subspaces, $x_a$ and $x_d$. These two subspaces correspond to the finite zero and infinite zero structures of $\Sigma$, respectively. The new state space, input and output space of the decomposed system are described by the following set of equations:

\begin{equation}
\begin{aligned}
x &= \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u}, \\
\tilde{x} &= \begin{pmatrix} x_a \\ x_d \end{pmatrix}, \quad x_a \in \mathbb{R}^{n_a}, \quad x_d \in \mathbb{R}^{n_d}, \quad x_d = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n_d} \end{pmatrix},
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\dot{x}_a &= A_{aa} x_a + L_{ad} \tilde{y}, \\
\dot{x}_1 &= x_2, \quad \dot{y} = x_1, \\
\dot{x}_2 &= x_3, \\
& \vdots \\
\dot{x}_{n_d-1} &= x_{n_d}, \\
\dot{x}_{n_d} &= E_{da} x_a + E_{1} x_1 + E_{2} x_2 + \cdots + E_{n_d} x_{n_d} + \tilde{u}.
\end{aligned}
\end{equation}

Furthermore, $\lambda(A_{aa})$ contains all the system invariant zeros and $n_d$ is the relative degree of $\Sigma$. 

\[ \Sigma : \dot{x} = A \ x + B \ u, \quad y = C \ x, \quad (5.2.1) \]
Invariant zero dynamics

Infinite zero structure

Note: the signal given by the double-edged arrow is a linear combination of the states.

Figure 5.2.1: Interpretation of structural decomposition of a SISO system.
Example 5.2.1. Consider a SISO system $\Sigma$ characterized by (5.2.1) with

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad (5.2.36)$$

and

$$C = \begin{bmatrix} 0 & 3 & -2 & 0 \end{bmatrix}. \quad (5.2.37)$$

The structural decomposition of $\Sigma$ proceeds as follows:

1. Differentiating the system output.

It involves the following sub-steps.

(a) First, we have

$$\dot{y} = C \dot{x} = CAx + CBu = \begin{bmatrix} -2 & -1 & 0 & 1 \end{bmatrix} x + 0 \cdot u.$$

(b) Since $CB = 0$, we compute

$$\ddot{y} = CAx + CABu = \begin{bmatrix} 1 & -1 & -3 & 1 \end{bmatrix} x + 0 \cdot u.$$

(c) Since $CAB = 0$, we continue on computing

$$y^{(3)} = CAx + CA^2 Bu = -\begin{bmatrix} 8 & 10 & 12 & 17 \end{bmatrix} x - 6 \cdot u.$$

We move to the next step as $CA^2 B \neq 0$. 
2. Constructing a preliminary state transformation.

Let $Z_0$ be a vector such that

\[
Z = \begin{bmatrix} Z_0 \\ C \\ CA \\ CA^2 \end{bmatrix},
\]  

(5.2.38)

is nonsingular. Then, define a new set of state variables $\bar{x}$,

\[
\bar{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} : = Zx = \begin{bmatrix} Z_0 \\ C \\ CA \\ CA^2 \end{bmatrix} x = \begin{pmatrix} Z_0x \\ y \\ \dot{y} \\ \ddot{y} \end{pmatrix}.
\]  

(5.2.39)

It is simple to verify that $Z$ with $Z_0 = [1 \ 0 \ 0 \ 0]$ is a nonsingular matrix. Furthermore,

\[
\dot{x}_0 = 8x_0 + x_1 + \frac{8}{3}x_2 - \frac{5}{3}x_3 + u,
\]  

(5.2.40)

\[
\dot{x}_1 = x_2,
\]  

(5.2.41)

\[
\dot{x}_2 = x_3,
\]  

(5.2.42)

\[
\dot{x}_3 = -72x_0 - 9x_1 - 27x_2 + 10x_3 - 6u.
\]  

(5.2.43)
3. **Eliminating $u$ in (5.2.40).**

Equation (5.2.43) implies that

$$ u = -12x_0 - \frac{3}{2}x_1 - \frac{9}{2}x_2 + \frac{5}{3}x_3 - \frac{1}{6}\dot{x}_3. \quad (5.2.44) $$

Substituting this into (5.2.40), we obtain

$$ \dot{x}_0 = -4x_0 - \frac{1}{2}x_1 - \frac{11}{6}x_2 - \frac{1}{6}\dot{x}_3. \quad (5.2.45) $$

We have eliminated $u$ in $\dot{x}_0$. Unfortunately, we have also introduced an additional $\dot{x}_3$ in (5.2.45).

4. **Eliminating $\dot{x}_3$ in (5.2.45).**

Define a new variable $\tilde{x}_0$ as

$$ \tilde{x}_0 := x_0 + \frac{1}{6}x_3. \quad (5.2.46) $$

We have

$$ \dot{\tilde{x}}_0 = -4\tilde{x}_0 - \frac{1}{2}x_1 - \frac{11}{6}x_2 + \frac{2}{3}x_3, \quad (5.2.47) $$

and

$$ \dot{x}_3 = -72\tilde{x}_0 - 9x_1 - 27x_2 + 22x_3 - 6u. \quad (5.2.48) $$
5. Eliminating $x_2$ and $x_3$ in (5.2.47).

This step involves two sub-steps.

(a) Letting

$$\tilde{x}_{0,1} := \tilde{x}_0 - \frac{2}{3}x_2,$$

we have

$$\dot{\tilde{x}}_{0,1} = -4\tilde{x}_{0,1} - \frac{1}{2}x_1 - \frac{9}{2}x_2,$$

and

$$\dot{x}_3 = -72\tilde{x}_{0,1} - 9x_1 - 75x_2 + 22x_3 - 6u.$$  \hfill (5.2.50)

(b) Letting

$$\tilde{x}_{0,2} := \tilde{x}_{0,1} + \frac{9}{2}x_1,$$

we have

$$\dot{\tilde{x}}_{0,2} = -4\tilde{x}_{0,2} + \frac{35}{2}x_1,$$

and

$$\dot{x}_3 = -72\tilde{x}_{0,2} + 315x_1 - 75x_2 + 22x_3 - 6u.$$  \hfill (5.2.54)
6. Composing the nonsingular state, output and input transformations.

Let

$$x_a := \tilde{x}_{0,2}$$

(5.2.55)

or equivalently let

$$x = \Gamma_s \tilde{x} = \Gamma_s \begin{pmatrix} x_a \\ x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

(5.2.56)

with

$$\Gamma_s = \left\{ \begin{bmatrix} 1 & 9/2 & -2/3 & 1/6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ -2 & -1 & 0 & 1 \\ 1 & -1 & -3 & 1 \end{bmatrix} \right\}^{-1}.$$  

(5.2.57)

Also, let

$$u = \Gamma_i \tilde{u} = -\frac{1}{6} \tilde{u}, \quad y = \Gamma_o \tilde{y} = 1 \cdot \tilde{y}.$$  

(5.2.58)
Finally, we obtain the dynamic equations of the transformed system,

\[
\begin{align*}
\dot{x}_a &= -4x_a + \frac{35}{2}x_1, \\
\dot{x}_1 &= x_2, \quad \ddot{y} = x_1, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= -72x_a + 315x_1 - 75x_2 + 22x_3 + \tilde{u},
\end{align*}
\] (5.2.59) (5.2.60) (5.2.61) (5.2.62)

An invariant zero at \(-4\)

An infinite zero of order 3 = relative degree
5.3 Strictly Proper Systems

Next, we consider a general strictly proper linear system $\Sigma$ characterized by

$$\begin{align*}
\dot{x} &= A \ x + B \ u, \\
y &= C \ x,
\end{align*} \tag{5.3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output. Without loss of generality, we assume that both $B$ and $C$ are of full rank. We have the following structural or special coordinate basis decomposition of $\Sigma$.

**Theorem 5.3.1.** Consider the strictly proper system $\Sigma$ characterized by (5.3.1). There exist a nonsingular state transformation, $\Gamma_s \in \mathbb{R}^{n \times n}$, a nonsingular output transformation, $\Gamma_o \in \mathbb{R}^{p \times p}$, and a nonsingular input transformation, $\Gamma_i \in \mathbb{R}^{m \times m}$, that will reveal all the structural properties of $\Sigma$. More specifically, we have

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u}, \tag{5.3.2}$$

with the new state variables

$$\tilde{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \quad x_a \in \mathbb{R}^{n_a}, \quad x_b \in \mathbb{R}^{n_b}, \quad x_c \in \mathbb{R}^{n_c}, \quad x_d \in \mathbb{R}^{n_d}, \tag{5.3.3}$$
the new output variables

\[ \tilde{y} = \begin{pmatrix} y_d \\ y_b \end{pmatrix}, \quad y_d \in \mathbb{R}^{md}, \quad y_b \in \mathbb{R}^{pb}, \] (5.3.4)

and the new input variables

\[ \tilde{u} = \begin{pmatrix} u_d \\ u_c \end{pmatrix}, \quad u_d \in \mathbb{R}^{md}, \quad u_c \in \mathbb{R}^{mc}. \] (5.3.5)

Further, the state variable \( x_d \) can be decomposed as:

\[ x_d = \begin{pmatrix} x_{d,1} \\ x_{d,2} \\ \vdots \\ x_{d,md} \end{pmatrix}, \quad y_d = \begin{pmatrix} y_{d,1} \\ y_{d,2} \\ \vdots \\ y_{d,md} \end{pmatrix}, \quad u_d = \begin{pmatrix} u_{d,1} \\ u_{d,2} \\ \vdots \\ u_{d,md} \end{pmatrix}, \] (5.3.6)

\[ x_{d,i} \in \mathbb{R}^{q_i}, \quad x_{d,i} = \begin{pmatrix} x_{d,i,1} \\ x_{d,i,1} \\ \vdots \\ x_{d,i,q_i} \end{pmatrix}, \quad i = 1, 2, \ldots, md, \] (5.3.7)
with $q_1 \leq q_2 \leq \cdots \leq q_{md}$. The state variable $x_b$ can be decomposed as

$$x_b = \begin{pmatrix} x_{b,1} \\ x_{b,2} \\ \vdots \\ x_{b,p_b} \end{pmatrix}, \quad y_b = \begin{pmatrix} y_{b,1} \\ y_{b,2} \\ \vdots \\ y_{b,p_b} \end{pmatrix}, \quad (5.3.8)$$

$$x_{b,i} \in \mathbb{R}^{l_i}, \quad x_{b,i} = \begin{pmatrix} x_{b,i,1} \\ x_{b,i,2} \\ \vdots \\ x_{b,i,l_i} \end{pmatrix}, \quad i = 1, 2, \ldots, p_b, \quad (5.3.9)$$

with $l_1 \leq l_2 \leq \cdots \leq l_{p_b}$. And finally, the state variable $x_c$ can be decomposed as

$$x_c = \begin{pmatrix} x_{c,1} \\ x_{c,2} \\ \vdots \\ x_{c,m_c} \end{pmatrix}, \quad u_c = \begin{pmatrix} u_{c,1} \\ u_{c,2} \\ \vdots \\ u_{c,m_c} \end{pmatrix}, \quad (5.3.10)$$

$$x_{c,i} \in \mathbb{R}^{r_i}, \quad x_{c,i} = \begin{pmatrix} x_{c,i,1} \\ x_{c,i,2} \\ \vdots \\ x_{c,i,r_i} \end{pmatrix}, \quad i = 1, 2, \ldots, m_c, \quad (5.3.11)$$

with $r_1 \leq r_2 \leq \cdots \leq r_{m_c}$. 
The decomposed system can be expressed in the following dynamical equations:

\[ \dot{x}_a = A_{aa}x_a + L_{ab}y_b + L_{ad}y_d, \]  

(5.3.12)

for each subsystem \( x_{b,i} \), \( i = 1, 2, \ldots, p_b \),

\[ \begin{align*}
\dot{x}_{b,i,1} &= x_{b,i,2} + L_{bd,i,1}y_b + L_{b,i,1}y_d, \quad y_b,i = x_{b,i,1}, \\
\dot{x}_{b,i,2} &= x_{b,i,3} + L_{bd,i,2}y_b + L_{b,i,2}y_d, \\
& \vdots \\
\dot{x}_{b,i,l_i} &= L_{bd,i,l_i}y_b + L_{b,i,l_i}y_d,
\end{align*} \]  

(5.3.13)

(5.3.14)

(5.3.15)

for each subsystem \( x_{c,i} \), \( i = 1, 2, \ldots, m_c \),

\[ \begin{align*}
\dot{x}_{c,i,1} &= x_{c,i,2} + L_{cb,i,1}y_b + L_{cd,i,1}y_d, \\
& \vdots \\
\dot{x}_{c,i,r_i-1} &= x_{c,i,r_i-1} + L_{cb,i,r_i-1}y_b + L_{cd,i,r_i-1}y_d, \\
\dot{x}_{c,i,r_i} &= A_{c,i,a}x_a + A_{c,i,c}x_c + L_{cb,i,r_i}y_b + L_{cd,i,r_i}y_d + u_{c,i},
\end{align*} \]  

(5.3.16)

(5.3.17)

(5.3.18)
and finally, for each subsystem $x_{d,i}, i = 1, 2, \ldots, m_d$,

\begin{align}
\dot{x}_{d,i,1} &= x_{d,i,2} + L_{d,i,1}y_d, \quad y_{d,i} = x_{d,i,1}, \\
\dot{x}_{d,i,2} &= x_{d,i,3} + L_{d,i,2}y_d, \\
&\vdots \\
\dot{x}_{d,i,q_i} &= A_{d,i,a}x_a + A_{d,i,c}x_c + A_{d,i,b}x_b + A_{d,i,d}x_d + u_{d,i},
\end{align}  \tag{5.3.19-5.3.21}

where $A_{aa}, L_{ab}, \ldots, A_{d,i,d}$ are constant matrices of appropriate dimensions.

infinite zero structure

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$x_a$ – the subsystem without direct input and output:

![Diagram of $x_a$]

invariant zero

$x_{b,i}$ – the chain of integrators without a direct input:

![Diagram of $x_{b,i}$]

left invertibility structure

$x_{c,i}$ – the chain of integrators without a direct output:

![Diagram of $x_{c,i}$]

right invertibility structure

$x_{d,i}$ – the chain of integrators with direct input and output:

![Diagram of $x_{d,i}$]

infinite zero structure
Example 5.3.1. Consider a strictly proper system $\Sigma$ characterized by (5.3.1) with

$$A = \begin{bmatrix}
0 & 0 & 2 & -1 & 2 & 0 & -1 & 2 & 0 & -1 & 0 & 2 & 2 \\
0 & 2 & 4 & -5 & 3 & 2 & -4 & 3 & 2 & -4 & 0 & 5 & 0 \\
0 & -1 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 3 & -2 & 0 & 3 & -3 & 0 & 3 & 1 & -1 & 0 \\
0 & 2 & 2 & 0 & -2 & 2 & 1 & -3 & 2 & 1 & 1 & 1 & -2 \\
0 & 3 & 3 & -1 & -2 & 3 & 0 & -2 & 3 & 0 & 2 & 2 & -3 \\
0 & 3 & 3 & -1 & -2 & 3 & -1 & -1 & 3 & 0 & 1 & 3 & -3 \\
0 & 3 & 3 & -1 & -2 & 3 & -1 & 0 & 3 & 0 & 1 & 4 & -3 \\
0 & 2 & 2 & 1 & -1 & 2 & 0 & 0 & 2 & 1 & 1 & 3 & -1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 0 \\
0 & 0 & -2 & 4 & -2 & 0 & 2 & -2 & 0 & 2 & 1 & -2 & 0 \\
0 & -1 & -3 & 7 & -3 & -1 & 4 & -3 & -1 & 4 & 2 & -4 & 1 \\
-1 & 0 & 0 & 1 & 1 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 2
\end{bmatrix},$$

$$B = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 2 & 2 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & -1 & 2 & -1 \\
0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0
\end{bmatrix}.
The required state, input and output transformations...

\[
\Gamma_s = \begin{bmatrix}
-1 & -1 & 1 & -1 & 3 & -3 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\
-1 & 0 & 5 & 0 & 10 & 0 & 0 & 0 & 0 & 3 & 2 & 1 \\
0 & -1 & -2 & 0 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 2 & 0 & 4 & 0 & 1 & 0 & 1 & 0 & 4 & 1 \\
0 & -2 & 2 & 0 & 5 & 0 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\
1 & -3 & 7 & 0 & 14 & 1 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\
0 & -2 & 4 & 0 & 9 & 0 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\
1 & -1 & 5 & 0 & 10 & 0 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\
1 & 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 2 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 & 6 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\
\end{bmatrix}
\]

\[
\Gamma_i = \begin{bmatrix}
0 & 1 & -1 & 0 \\
1 & -1 & 1 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{bmatrix}, \quad \Gamma_o = \begin{bmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 \\
\end{bmatrix}
\]
The transformed system \( (\tilde{A}, \tilde{B}, \tilde{C}) = (\Gamma_s^{-1}A\Gamma_s, \Gamma_s^{-1}B\Gamma_i, \Gamma_o^{-1}C\Gamma_s) \)

\[
\tilde{A} = \begin{bmatrix}
-2 & 0 & 6 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & -2 & 0 & -5 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
5 & -1 & -1 & 0 & -3 & 0 & 1 & 1 & 2 & 5 \\
2 & -2 & 12 & 6 & 23 & 8 & 1 & 1 & 2 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & -2 & 8 & 3 & 15 & 4 & 1 & 1 & 2 & 1 \\
\end{bmatrix}
\]

\[
\tilde{B} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\tilde{C} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
F \quad \text{and} \quad K
\]

\[
\bar{A} = \tilde{A} + \tilde{B}F + K\tilde{C}
\]
The essential structures of the system...

\[ \lambda(A_{aa}) = \{-2, 1\} \]

\[ S_L^*(\Sigma) = \{2, 2\} \]

\[ S_R^*(\Sigma) = \{1, 2\} \]

\[ S_\infty^*(\Sigma) = \{1, 3\} \]
5.4 Nonstrictly Proper Systems

We now present in this section the structural decomposition or the special coordinate basis of general nonstrictly proper multivariable systems. We will also present all the structural properties of such a decomposition with rigorous proofs. To be specific, we consider the following nonstrictly proper system $\Sigma$ characterized by

$$
\begin{align*}
\dot{x} &= Ax + Bu, \\
y &=Cx + Du,
\end{align*}
$$

(5.4.1)

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of $\Sigma$. Without loss of generality, we assume that both $[B' \quad D']$ and $[C \quad D]$ are of full rank.

The structural decomposition or the special coordinate basis of nonstrictly proper systems follows fairly closely from that of strictly proper systems given in Section 5.3. However, in many applications, it is not necessary to decompose the subsystems $x_b$ and $x_c$ into chains of integrators. On the other hand, in many situations, it is necessary to further separate $x_a$, the subsystem related to the invariant zero dynamics of the given system, into subspaces corresponding to the stable, marginally stable (or marginally unstable) and unstable zero dynamics.
For future use, we rewrite the structural decomposition of $\Sigma$ in a more compact form:

$$\hat{A} = \Gamma_s^{-1} A \Gamma_s = A_s + B_0 C_0 = \begin{bmatrix}
A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\
0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\
0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\
0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\
B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\
B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd}
\end{bmatrix} + B_0 C_0$$

$$\hat{B} = \Gamma_s^{-1} B \Gamma_i = \begin{bmatrix} B_0 & B_s \end{bmatrix} = \begin{bmatrix}
P_{0a}^- & 0 & 0 & B_{0a}^- & 0 & 0 \\
P_{0a}^0 & 0 & 0 & B_{0a}^0 & 0 & 0 \\
P_{0a}^+ & 0 & 0 & B_{0a}^+ & 0 & 0 \\
B_{0b} & 0 & 0 & B_{0b} & 0 & B_c \\
B_{0c} & 0 & B_c & B_{0c} & 0 & B_c \\
B_{0d} & B_{d} & 0 & B_{0d} & B_{d} & 0 \\
\end{bmatrix}, \quad \hat{D} = \Gamma_i^{-1} D \Gamma_i = D_s = \begin{bmatrix}
I_{m_0} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}$$

$$\hat{C} = \Gamma_i^{-1} C \Gamma_s = \begin{bmatrix} C_0 \\ C_s \end{bmatrix} = \begin{bmatrix}
C_{0a}^- & C_{0a}^0 & C_{0a}^+ & C_{0b} & C_{0c} & C_{0d} \\
0 & 0 & 0 & 0 & 0 & C_d \\
0 & 0 & 0 & C_b & 0 & 0 \\
\end{bmatrix}$$
**Note:** In the compact SCB form...

\[ A_{dd} = A_{dd}^* + B_d E_{dd} + L_{dd} C_d, \]

for some constant matrices \( L_{dd} \) and \( E_{dd} \) of appropriate dimensions, and

\[ A_{dd}^* = \text{blkdiag}\left\{ A_{q_1}, A_{q_2}, \ldots, A_{q_{m_d}} \right\}, \]

\[ B_d = \text{blkdiag}\left\{ B_{q_1}, B_{q_2}, \ldots, B_{q_{m_d}} \right\}, \quad C_d = \text{blkdiag}\left\{ C_{q_1}, C_{q_2}, \ldots, C_{q_{m_d}} \right\} \]

\[
A_{q_i} = \begin{bmatrix} 0 & I_{q_i-1} \\ 0 & 0 \end{bmatrix}, \quad B_{q_i} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{q_i} = [1, 0, \ldots, 0].
\]

Moreover, \( \lambda(A_{aa}^-) \subset \mathbb{C}^- \), \( \lambda(A_{aa}^0) \subset \mathbb{C}^0 \)

and \( \lambda(A_{aa}^+) \subset \mathbb{C}^+ \). Also, \((A_{cc}, B_c)\) is controllable and \((A_{bb}, C_b)\) is observable.
For a strictly proper system, it has the following form...

\[
\tilde{A} = \Gamma_s^{-1} A \Gamma_s = 
\begin{bmatrix}
A_{aa} & 0 & 0 & L_{ab} C_b & 0 & L_{ad} C_d \\
0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\
0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\
0 & 0 & 0 & A_{bb} & 0 & L_{bd} C_d \\
B_c E_{ca}^- & B_c E_{ca}^0 & B_c E_{ca}^+ & L_{cb} C_b & A_{cc} & L_{cd} C_d \\
B_d E_{da}^- & B_d E_{da}^0 & B_d E_{da}^+ & B_d E_{db} & B_d E_{dc} & A_{dd}
\end{bmatrix}
\]

\[
\tilde{B} = \Gamma_s^{-1} B \Gamma_i = 
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & B_c \\
B_d & 0
\end{bmatrix}
\]

\[
\tilde{C} = \Gamma_o^{-1} C \Gamma_s = 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & C_d \\
0 & 0 & 0 & C_b & 0 & 0
\end{bmatrix}
\]
What can we do with a state feedback gain?...

\[
\tilde{A} + \tilde{B} \tilde{F} = \begin{bmatrix}
A_{aa}^- & 0 & 0 & L_{ab}^- C_b & 0 & L_{ad}^- C_d \\
0 & A_{aa}^0 & 0 & L_{ab}^0 C_b & 0 & L_{ad}^0 C_d \\
0 & 0 & A_{aa}^+ & L_{ab}^+ C_b & 0 & L_{ad}^+ C_d \\
0 & 0 & 0 & A_{bb} & 0 & A_{cc} \\
0 & 0 & 0 & 0 & A_{dd} & 0 \\
0 & 0 & 0 & 0 & 0 & B_c \\
\end{bmatrix}
\]

\[
\tilde{B} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
B_d \\
0 \\
\end{bmatrix}
\]

\[
\tilde{C} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & C_{cd} \\
0 & 0 & 0 & 0 & C_b \\
\end{bmatrix}
\]

\[\mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \] is a bad subspace for the state feedback control...

How about observer design?...

...duality...

\[\mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c \] is a bad subspace for observer design...  

...D.I.Y...
Property 5.4.1. The given system $\Sigma$ is observable (detectable) if and only if the pair $(A_{\text{obs}}, C_{\text{obs}})$ is observable (detectable), where

$$A_{\text{obs}} := \begin{bmatrix} A_{aa} & 0 \\ B_c E_{ca} & A_{cc} \end{bmatrix}, \quad C_{\text{obs}} := \begin{bmatrix} C_{0a} & C_{0c} \\ E_{da} & E_{dc} \end{bmatrix},$$

(5.4.28)

and where

$$A_{aa} := \begin{bmatrix} A_{aa}^- & 0 & 0 \\ 0 & A_{aa}^0 & 0 \\ 0 & 0 & A_{aa}^+ \end{bmatrix}, \quad C_{0a} := [C_{0a}^-, C_{0a}^0, C_{0a}^+],$$

(5.4.29)

$$E_{da} := [E_{da}^-, E_{da}^0, E_{da}^+], \quad E_{ca} := [E_{ca}^-, E_{ca}^0, E_{ca}^+].$$

(5.4.30)

Also, define

$$A_{\text{con}} := \begin{bmatrix} A_{aa} & L_{ab} C_b \\ 0 & A_{bb} \end{bmatrix}, \quad B_{\text{con}} := \begin{bmatrix} B_{0a} & L_{ad} \\ B_{0b} & L_{bd} \end{bmatrix},$$

(5.4.31)

$$B_{0a} := \begin{bmatrix} B_{0a}^- \\ B_{0a}^0 \\ B_{0a}^+ \end{bmatrix}, \quad L_{ab} := \begin{bmatrix} L_{ab}^- \\ L_{ab}^0 \\ L_{ab}^+ \end{bmatrix}, \quad L_{ad} := \begin{bmatrix} L_{ad}^- \\ L_{ad}^0 \\ L_{ad}^+ \end{bmatrix}.$$

(5.4.32)

Similarly, $\Sigma$ is controllable (stabilizable) if and only if the pair $(A_{\text{con}}, B_{\text{con}})$ is controllable (stabilizable).
Property 5.4.2. The structural decomposition also shows explicitly the invariant zeros and the normal rank of $\Sigma$. To be more specific, we have the following properties:

1. The normal rank of $H(s)$ is equal to $m_0 + m_d$.

2. Invariant zeros of $\Sigma$ are the eigenvalues of $A_{aa}$, which are the unions of the eigenvalues of $A_{aa}^-$, $A_{aa}^0$ and $A_{aa}^+$. 

Obviously, $\Sigma$ is of minimum phase if and only if $n_a^0 + n_a^+ = 0$. Otherwise, it is of nonminimum phase.
Property 5.4.4. $\Sigma$ has $m_0 = \text{rank}(D)$ infinite zeros of order 0. The infinite zero structure (of order greater than 0) of $\Sigma$ is given by

$$S_\infty^*(\Sigma) = \{q_1, q_2, \ldots, q_{m_0}\}. \quad (5.4.36)$$

That is, each $q_i$ corresponds to an infinite zero of order $q_i$. In particular, for a strictly proper SISO system $\Sigma$, we have $S_\infty^*(\Sigma) = \{q_1\}$, where $q_1$ is the relative degree of $\Sigma$. The given system $\Sigma$ is said to be of uniform rank if either $m_0 = 0$ and $q_1 = q_2 = \cdots = q_{m_0}$, or $m_0 \neq 0$ and $S_\infty^*(\Sigma) = \emptyset$.

Property 5.4.5. The given system $\Sigma$ is right invertible if and only if $x_b$ (and hence $y_b$) are nonexistent, left invertible if and only if $x_c$ (and hence $u_c$) are nonexistent, and invertible if and only if both $x_b$ and $x_c$ are nonexistent. Moreover, $\Sigma$ is degenerate if and only if both $x_b$ and $x_c$ are present.
The structural decomposition decomposes the state space of $\Sigma$ into several distinct parts. In fact, the state space $\mathcal{X}$ is decomposed as

$$\mathcal{X} = \mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_b \oplus \mathcal{X}_c \oplus \mathcal{X}_d.$$  \hspace{1cm} (5.4.37)

Here $\mathcal{X}_a^-$ is related to the stable invariant zeros, i.e., the eigenvalues of $A_{aa}$ are the stable invariant zeros of $\Sigma$. Similarly, $\mathcal{X}_a^0$ and $\mathcal{X}_a^+$ are respectively related to the invariant zeros of $\Sigma$ located in the marginally stable and unstable regions. On the other hand, $\mathcal{X}_b$ is related to the right invertibility, i.e., the system is right invertible if and only if $\mathcal{X}_b = \{0\}$, while $\mathcal{X}_c$ is related to left invertibility, i.e., the system is left invertible if and only if $\mathcal{X}_c = \{0\}$. Finally, $\mathcal{X}_d$ is related to zeros of $\Sigma$ at infinity.

There are interconnections between the subsystems generated by the structural decomposition and various invariant geometric subspaces. The following properties show these interconnections.
Property 5.4.6. The geometric subspaces defined in Definitions 3.7.2 and 3.7.4 are given by:

1. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_c$ spans $\mathcal{V}^-(\Sigma)$.
2. $\mathcal{X}_a^+ \oplus \mathcal{X}_c$ spans $\mathcal{V}^+(\Sigma)$.
3. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c$ spans $\mathcal{V}^*(\Sigma)$.
4. $\mathcal{X}_a^+ \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ spans $\mathcal{S}^-(\Sigma)$.
5. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ spans $\mathcal{S}^+(\Sigma)$.
6. $\mathcal{X}_c \oplus \mathcal{X}_d$ spans $\mathcal{S}^*(\Sigma)$.
7. $\mathcal{X}_c$ spans $\mathcal{R}^*(\Sigma)$.
8. $\mathcal{X}_a^- \oplus \mathcal{X}_a^0 \oplus \mathcal{X}_a^+ \oplus \mathcal{X}_c \oplus \mathcal{X}_d$ spans $\mathcal{N}^*(\Sigma)$. 
Property 5.4.7. The geometric subspaces defined in Definition 3.7.5, i.e., $S_\lambda(\Sigma)$ and $V_\lambda(\Sigma)$, can be computed as follows:

\[
S_\lambda(\Sigma) = \text{im} \left\{ \Gamma_s \begin{bmatrix}
\lambda I - A_{aa} & 0 & 0 & 0 \\
0 & Y_{b\lambda} & 0 & 0 \\
0 & 0 & I_{n_c} & 0 \\
0 & 0 & 0 & I_{n_d}
\end{bmatrix} \right\},
\]

where

\[
im \{Y_{b\lambda}\} = \ker \left[ C_b(A_{bb} + K_b C_b - \lambda I)^{-1} \right],
\]

and where $K_b$ is any matrix of appropriate dimensions and subject to the constraint that $A_{bb} + K_b C_b$ has no eigenvalue at $\lambda$. We note that such a $K_b$ always exists as $(A_{bb}, C_b)$ is observable.

\[
V_\lambda(\Sigma) = \text{im} \left\{ \Gamma_s \begin{bmatrix}
X_{a\lambda} & 0 \\
0 & 0 \\
0 & 0 \\
0 & X_{c\lambda}
\end{bmatrix} \right\},
\]

where $X_{a\lambda}$ is a matrix whose columns form a basis for the subspace,

\[
\left\{ \zeta_a \in \mathbb{C}^{n_a} \mid (\lambda I - A_{aa})\zeta_a = 0 \right\},
\]

and

\[
X_{c\lambda} := \left( A_{cc} + B_c F_c - \lambda I \right)^{-1} B_c,
\]

with $F_c$ being any matrix of appropriate dimensions and subject to the constraint that $A_{cc} + B_c F_c$ has no eigenvalue at $\lambda$. Again, we note that the existence of such an $F_c$ is guaranteed by the controllability of $(A_{cc}, B_c)$. 
Example 5.4.1. Let us reconsider the system $\Sigma$ of Example 5.3.1, i.e., consider a matrix quadruple $(A, B, C, D)$ with $(A, B, C)$ being the same as those given in Example 5.3.1 and $D = 0$. All the necessary transformations required to transform the given system into the special coordinate basis have already been obtained in Example 5.3.1.

$$\nu^-(\Sigma) = \text{im} \begin{bmatrix} -2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 6 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \nu^+(\Sigma) = \text{im} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\nu^*(\Sigma) = \text{im} \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 6 & 3 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 4 & 1 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{R}^*(\Sigma) = \text{im} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
$\nu_\lambda(\Sigma) = \text{im} \left\{ \begin{bmatrix} -2 & 0 & 0 \\ -3 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & 1 & 1 \\ 2 & 1 & 0 \\ 6 & 1 & 2 \\ 2 & 0 & 1 \\ 4 & 0 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}, \quad \lambda = -2,$

$\nu_\lambda(\Sigma) = \text{im} \left\{ \begin{bmatrix} -2 & 4 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ -3 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & -2 & 1 & 0 & 1 & 0 & 4 & 1 & 0 \\ 2 & 2 & -1 & 1 & 1 & 1 & 0 & 6 & 2 & 0 \\ 6 & -2 & -8 & 1 & 1 & 2 & -1 & 11 & 4 & 1 \\ 2 & 2 & -3 & 0 & 1 & 1 & 0 & 7 & 3 & 1 \\ 4 & 0 & -5 & 0 & 0 & 1 & 0 & 6 & 2 & 1 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \end{bmatrix} \right\}, \quad \lambda = 1.$

We note that $\lambda = -2$ and $\lambda = 1$ correspond respectively to the stable and the unstable invariant zeros of $\Sigma.$
**Exercise 5.1.** Compute a special coordinate basis for the SISO system

\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
x \\ 1 \\ 1 \\ 1 \\
\end{bmatrix} + \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
\end{bmatrix} u, \quad y = \begin{bmatrix}
1 & -1 & 1 & -1 \\
\end{bmatrix} \begin{bmatrix}
x \\
\end{bmatrix}.
\]

Identify the invariant zeros and the relative degree of the given system.

**Exercise 5.2.** Utilize the properties of the special coordinate basis to construct a fourth order controllable and observable SISO system, \( \Sigma \), for each of the following five cases:

(a) \( \Sigma \) has no invariant zeros and has a relative degree of 4.
(b) \( \Sigma \) has one invariant zero at \( \{1\} \) and has a relative degree of 3.
(c) \( \Sigma \) has two invariant zeros at \( \{1, 2\} \), and has a relative degree of 2.
(d) \( \Sigma \) has three invariant zeros at \( \{1, 2, 3\} \), and has a relative degree of 1.
(e) \( \Sigma \) has four invariant zeros at \( \{\pm j, \pm 1\} \), and has a relative degree of 0.
Exercise 5.5. Utilize the properties of the special coordinate basis to construct a fourth order invertible, controllable and observable MIMO system, $\Sigma$, for each of the following cases:

(a) $\Sigma$ is strictly proper, and has an infinite zero structure $S_{\infty}^* = \{1, 3\}$, which implies that $\Sigma$ is free of invariant zeros.

(b) $\Sigma$ is strictly proper, and has an infinite zero structure $S_{\infty}^* = \{2, 2\}$, which implies that $\Sigma$ is free of invariant zeros.

(c) $\Sigma$ is strictly proper, and has one invariant zero at $\{1\}$ and an infinite zero structure $S_{\infty}^* = \{1, 2\}$.

(d) $\Sigma$ is strictly proper, and has two invariant zeros at $\{\pm j\}$ and an infinite zero structure $S_{\infty}^* = \{1, 1\}$.

(e) $\Sigma$ is nonstrictly proper, and has three invariant zeros at $\{1, \pm j\}$ and an infinite zero structure $S_{\infty}^* = \{1\}$.

(f) $\Sigma$ is nonstrictly proper, and has four invariant zeros at $\{\pm 1, \pm j\}$ and no infinite zero of order higher than 0.
**Exercise 5.6.** Construct a third order strictly proper and right invertible system, $\Sigma$, with two inputs and one output, for each of the following cases:

(a) $\Sigma$ has an infinite zero of order 2, and has no invariant zeros.

(b) $\Sigma$ has an infinite zero of order 1, and has one invariant zero at $\{-1\}$.

Moreover, the obtained systems must be controllable and unobservable.

**Exercise 5.7.** Construct a third order strictly proper and left invertible system, $\Sigma$, with one input and two outputs, for each of the following cases:

(a) $\Sigma$ has an infinite zero of order 2, and has no invariant zeros.

(b) $\Sigma$ has an infinite zero of order 1, and has one invariant zero at $\{-1\}$.

Furthermore, the obtained systems must be uncontrollable and observable.

**Exercise 5.8.** Construct a second order system, $\Sigma$, which has the following properties: (i) $\Sigma$ is neither left nor right invertible; (ii) $\Sigma$ is uncontrollable and unobservable; (ii) $\Sigma$ is free of finite zeros and is free of infinite zeros of order higher than 0; and (iv) $\Sigma$ is nonstrictly proper with both $[C \ D]$ and $[B' \ D']$ being of full rank.
Exercise 5.11. Consider a SISO system, $\Sigma$, which is already in the SCB form as given in Theorem 5.2.1, i.e.,

\[
\begin{align*}
\dot{x}_a &= A_{aa} x_a + L_{ad} y, \\
\dot{x}_1 &= x_2, \quad y = x_1, \\
\dot{x}_2 &= x_3, \quad \ldots, \quad \dot{x}_{n_d-1} = x_{n_d}, \\
\dot{x}_{n_d} &= E_{da} x_a + E_1 x_1 + E_2 x_2 + \cdots + E_{n_d} x_{n_d} + u,
\end{align*}
\]

or in the matrix form:

\[
\begin{align*}
x &= A x + B u = \\
&= \begin{bmatrix} A_{aa} & L_{ad} & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
E_{da} & E_1 & E_2 & \cdots & E_{n_d} \end{bmatrix} \begin{bmatrix} x_a \\
x_1 \\
x_2 \\
\vdots \\
x_{n_d} \end{bmatrix} + \begin{bmatrix} 0 \\
0 \\
0 \\
\vdots \\
1 \end{bmatrix} u,
\end{align*}
\]

and

\[
y = C x = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix} x.
\]

Let

\[
\tilde{B} := B + \begin{bmatrix} K_a \\
0 \\
\vdots \\
0 \\
0 \end{bmatrix} = \begin{bmatrix} K_a \\
0 \\
\vdots \\
0 \\
1 \end{bmatrix}.
\]

Construct the special coordinate basis for the new system, $\tilde{\Sigma}$, characterized by $\dot{x} = A x + \tilde{B} u$, and $y = C x$. Show that $\Sigma$ and $\tilde{\Sigma}$ have the same relative degree. Also, show that the invariant zeros of $\tilde{\Sigma}$ are given by the eigenvalues of $\tilde{A}_{aa} := A_{aa} - K_a E_{da}$.