Solutions to Q.1 and Q.2

Q.1 Consider the electric circuit given in Fig. 1 below, in which $u(t)$ is a voltage source.

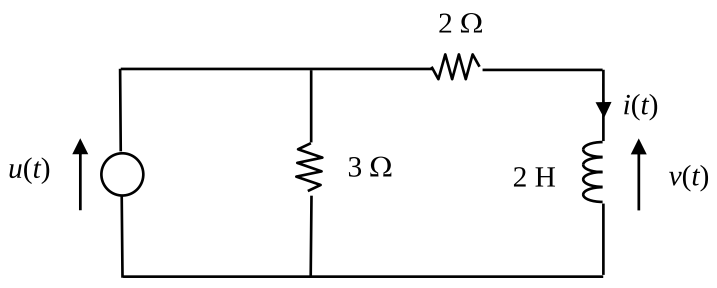

Fig. 1

(a) Derive a time-domain model for the electric circuit in terms of the inductor current, $i(t)$.

(5 marks)

Solution: By applying KVL to the outer loop of the circuit, we have

$$2\frac{di(t)}{dt} + 2i(t) = u(t)$$

(b) Show that the model obtained in Part (a) is time invariant.

(5 marks)

Solution: To show that the obtained system is time-invariant, we do the following test.

Step One: Suppose $i_1(t)$ is an output corresponding to an input signal, $u_1(t)$. We have

$$2\frac{di_1(t)}{dt} + 2i_1(t) = u_1(t) \quad \Rightarrow \quad 2\frac{di_1(t-t_0)}{d(t-t_0)} + 2i_1(t-t_0) = u_1(t-t_0)$$

$$\Rightarrow \quad 2\frac{di_1(t-t_0)}{dt} + 2i_1(t-t_0) = u_1(t-t_0)$$

Step Two: Let $u_2(t) = u_1(t-t_0)$. Then, $i_2(t) = i_1(t-t_0)$ has the following property:

$$2\frac{di_2(t)}{dt} + 2i_2(t) = 2\frac{di_1(t-t_0)}{dt} + 2i_1(t-t_0) = u_1(t-t_0) = u_2(t)$$

Clearly, $i_2(t)$ is an output produced by $u_2(t)$. Hence, the system is time-invariant.

(c) Show that the model obtained in Part (a) is linear.

(5 marks)

Solution: Assume that $i_1(t)$ and $i_2(t)$ are the outputs produced by $u_1(t)$ and $u_2(t)$, respectively, i.e.

$$2\frac{di_1(t)}{dt} + 2i_1(t) = u_1(t) \quad \& \quad 2\frac{di_2(t)}{dt} + 2i_2(t) = u_2(t)$$

If $i(t) = \alpha_1 i_1(t) + \alpha_2 i_2(t)$ is an output produced by the input $u(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$, then we can conclude that the system is linear. This can be verified as follows:

$$2\frac{di(t)}{dt} + 2i(t) = 2\frac{d}{dt}\left[\alpha_1 i_1(t) + \alpha_2 i_2(t)\right] + 2\left[\alpha_1 i_1(t) + \alpha_2 i_2(t)\right]$$
$$= \alpha_1\left[2\frac{di_1(t)}{dt} + 2i_1(t)\right] + \alpha_2\left[2\frac{di_2(t)}{dt} + 2i_2(t)\right] = \alpha_1 u_1(t) + \alpha_2 u_2(t) = u(t)$$

Thus, the system is indeed linear.

(d) Show that the system is BIBO stable.

(5 marks)

Solution: The characteristic polynomial of the system is given by

$$2s + 2 = 0 \quad \Rightarrow \quad s = -1$$

Its root is the LHP and thus the system is asymptotically stable and hence BIBO stable.

(e) Assume that the initial current of the inductor is 2 A and $u(t)$ = 2 V. Determine an explicit expression for the inductor current $i(t)$.

(5 marks)

Solution: Taking Laplace transform on both sides of the ODE, i.e.,

$$L\left\{2\frac{di(t)}{dt} + 2i(t)\right\} = L\{u(t)\} \quad \Rightarrow \quad 2\left[sI(s) - i(0)\right] + 2I(s) = \frac{2}{s} \quad \Rightarrow \quad I(s) = \frac{2/s + 4}{2s + 2} = \frac{2s+1}{s(s+1)}$$
$$\Rightarrow \quad i(t) = L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s} + \frac{1}{s+1}\right\} = 1 + e^{-t}$$

Q.2 The magnitude response of a typical second order system characterized by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

is given in Fig. 2 below.

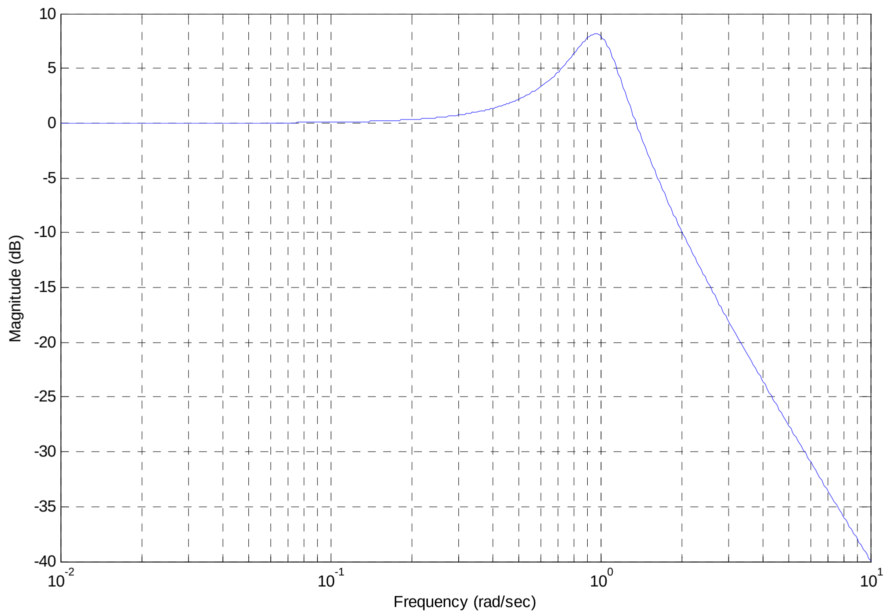

Fig. 2

(a) Find the DC gain, K, the damping ratio, ζ, and the natural frequency, ω_n, of the given system.

(10 marks)

Solution: It is simple to observe from the magnitude response that the static or DC gain is unity, i.e., $K = 1$. The corner frequency, which is also the natural frequency, of the magnitude response is 1 rad/sec, i.e., $\omega_n = 1$ rad/sec. The peak at the corner frequency is about 8 dB, which is corresponding to a damping ratio $\zeta = 0.2$. Thus, the transfer function is given by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{s^2 + 0.4s + 1}$$

(b) Given an input signal, $u(t) = \cos t$, find its corresponding steady-state output, $y(t)$.

(5 marks)

Solution: For the given input, we have ω = 1 rad/sec. Its corresponding frequency response is given by

$$H(j\omega)\Big|_{\omega=1} = \frac{1}{j^2 + j0.4 + 1} = -j2.5 = 2.5\angle -90°$$

Thus, the corresponding steady-state output is given by

$$y(t) = 2.5\cos(t - 90°)$$

(c) Find the overshoot, rise time, peak time and settling time of the unit step response of the system.

(5 marks)

Solution:

The overshoot is given by $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-3.14\times 0.2/\sqrt{1-0.2^2}} = 0.53 = 53\%$.

The rise time $t_r = \dfrac{1.8}{\omega_n} = 1.8$ sec.

The peak time $t_r = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{3.14}{\sqrt{1-0.2^2}} = 3.2$ sec.

The settling time $t_s = \dfrac{4}{\zeta\omega_n} = 20$ sec.

(d) Sketch the unit step response.

(5 marks)