Q.1 Consider the ball and beam system given in Fig. 1 below, in which θ is the system input. Assume that there is no friction in between the ball and the beam.

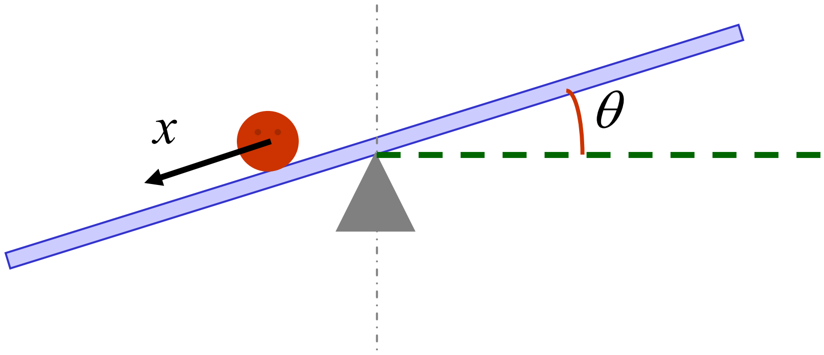


Fig. 1

(a) Derive a time-domain model for the system in terms of the displacement, $x(t)$.

(10 marks)

(b) Show that the model obtained in Part (a) is nonlinear.

(5 marks)

(c) Show that the model obtained in Part (a) is time invariant.

(5 marks)

(d) Show that the system is not BIBO stable.

(5 marks)

Q.2 Fig. 2 below shows the Bode plot and its asymptotes of a linear system.

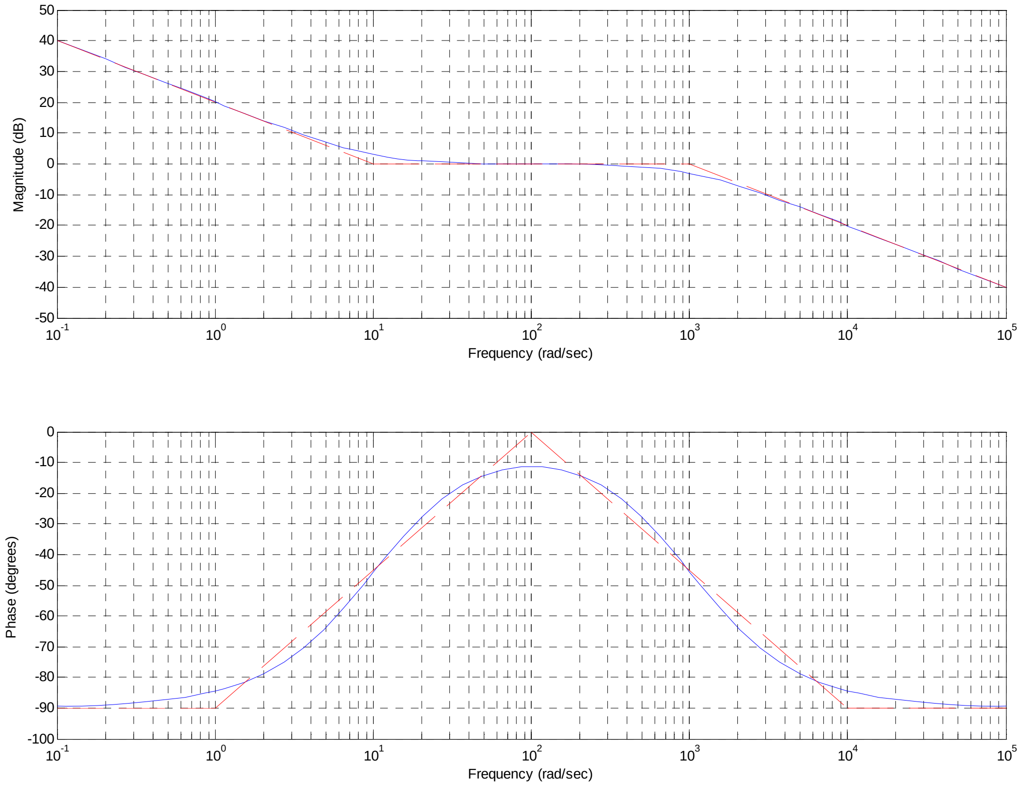


Fig. 2

(a) Obtain the system transfer function.

(10 marks)

(b) Given an input signal, $u(t) = \cos 100t$, find its corresponding steady-state output, $y(t)$.

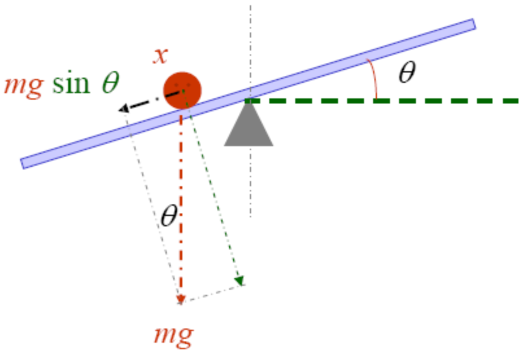
(8 marks)

(c) Find the unit step response of the system.

(7 marks)

Solution to Q.1:

(a) Since there is no friction on the surfaces, the only force acts on the system is the weight of the ball, i.e.



By Newton's law of motion, we have

$$F = ma \quad \Rightarrow \quad mg\sin\theta = ma = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = g\sin\theta$$

where g is the gravity constant, i.e., g = 9.8. Thus, the time-domain model of the system is

$$\ddot{x} = 9.8\sin\theta \quad \Leftrightarrow \quad \frac{d^2x(t)}{dt^2} = 9.8\sin\theta(t)$$

(b) Assume that the ball is initially stationary, i.e. $x(0) = 0$ and $\dot{x}(0) = 0$**.** Let θ_1=10° and let $x_1(t)$ be the corresponding solution, i.e.,

$$\frac{d^2x_1(t)}{dt^2} = 9.8\sin 10° = 1.7018 \quad \Rightarrow \quad x_1(t) = 0.8509t^2$$

Let $\theta = \alpha\, \theta_1 = 3 \times 10° = 30°$. However, it can be verified that the corresponding solution $x(t) \neq \alpha\, x_1(t)$**,** i.e.,

$$\frac{d^2x(t)}{dt^2} = 9.8\sin 30° = 4.9 \quad \Rightarrow \quad x(t) = 2.45t^2 \neq 3x_1(t) = 2.5527t^2$$

Thus, the system is nonlinear.

(c) The system is time-invariant. This can be verified by the following steps.

Step One: Suppose $x_1(t)$ is a solution corresponding to $\theta_1(t)$.

$$\frac{d^2x_1(t)}{dt^2} = 9.8\sin\theta_1(t) \quad \Rightarrow \quad \frac{d^2x_1(t-t_0)}{[d(t-t_0)]^2} = 9.8\sin\theta_1(t-t_0)$$

Step Two: Let $\theta_2(t) = \theta_1(t-t_0)$. Verify if $x_2(t) = x_1(t-t_0)$ is a solution to the system:

$$\frac{d^2x_2(t)}{dt^2} = \frac{d^2x_1(t-t_0)}{dt^2} = \frac{d^2x_1(t-t_0)}{[d(t-t_0)]^2} = 9.8\sin\theta_1(t-t_0) = 9.8\sin\theta_2(t)$$

which shows that $x_2(t)$ is indeed a solution corresponding to $\theta_2(t)$. By definition, the system is time-invariant.

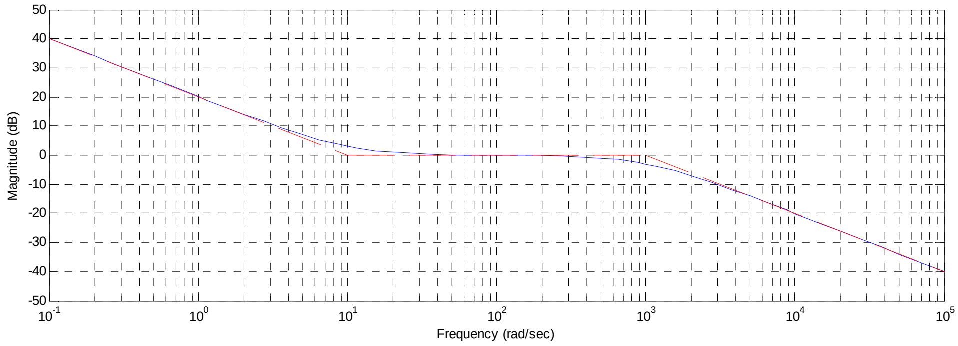
(d) The system is not BIBO stable. We show this by a specific example. Let the ball be initially stationary, i.e. $x(0) = 0$ and $\dot{x}(0) = 0$**,** and let $\theta = 1°$, which is bounded.

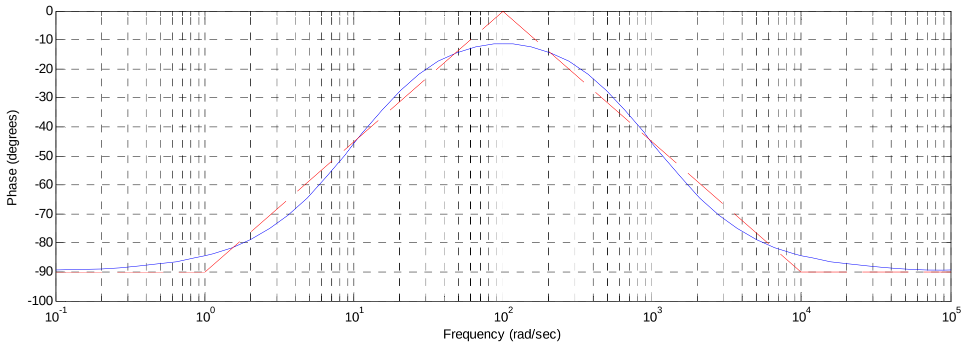
$$\frac{d^2x(t)}{dt^2} = 9.8\sin 1° = 0.171 \quad \Rightarrow \quad x(t) = 0.0855t^2 \to \infty \quad \text{as } t \to \infty$$

Clearly, $x(t)$ is unbounded. Thus, the system is BIBO unstable.

Solution to Q.2:

(a) From the Bode plot and the asymptotes,





we observe two corner frequencies, respectively, as 10 and 1000 rad/sec and one integrator. Thus, we have the transfer function of the system

$$G(s) = \frac{K\left(1+ {s}/{10}\right)}{s\left(1+ {s}/{1000}\right)}$$

The constant gain K can be determined by observing the magnitude response at $\omega = 1$ rad/sec in which the magnitude of $G(s)$ is given by

$$|G(j\omega)| \approx K = 20 \ dB = 10 \quad \Rightarrow \quad K = 10$$

Hence,

$$G(s) = \frac{10\left(1+ {s}/{10}\right)}{s\left(1+ {s}/{1000}\right)} = \frac{1000(s+10)}{s(s+1000)}$$

(b) For $\omega = 100$ rad/sec, the magnitude response is 0 dB (or 1) and the phase response is about −11.5 degrees. Thus, the steady-state output is given by

$$y(t) = \cos(100t - 11.5°)$$

(c) The unit step response of the system is given by

$$y(t) = L^{-1}\left[G(s)U(s)\right] = L^{-1}\left[\frac{1000(s+10)}{s(s+1000)}\frac{1}{s}\right]$$

$$= L^{-1}\left[\frac{0.99}{s} + \frac{10}{s^2} - \frac{0.99}{s+1000}\right] = 0.99 + 10t - 0.99e^{-1000t}$$