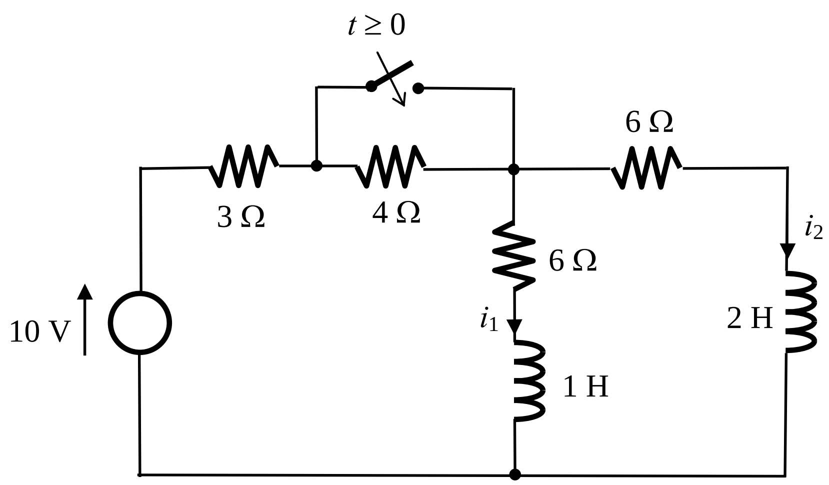
Q.1. In the electrical circuit given below, the switch has been in the position shown for a long time and is thrown to the other position for time $t \geq 0$.



(a) Determine the currents for both inductors for $t < 0$.

(4 marks)

(b) Determine the currents and voltages for both inductors just right after the switch is closed.

(4 marks)

(c) Derive the differential equation governing the circuit in terms of i_1.

(10 marks)

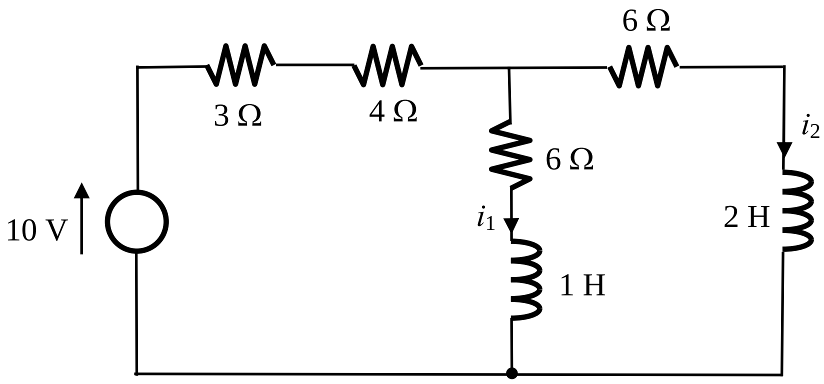
(d) Compute the roots of its characteristic polynomial.

(5 marks)

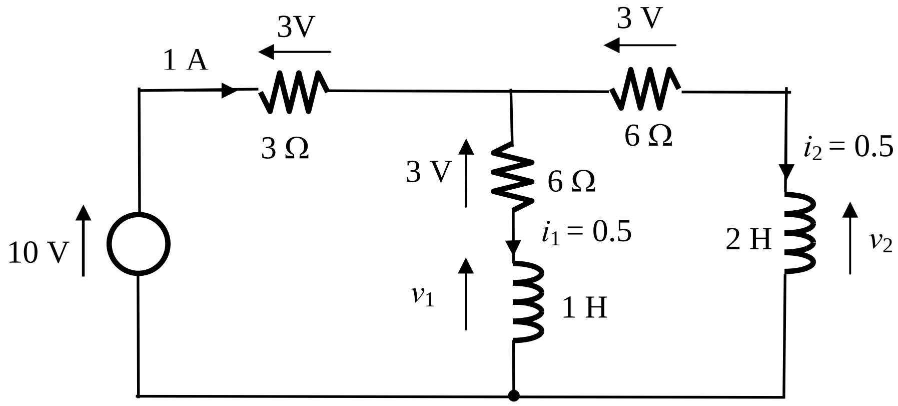
(e) Is the circuit over damped, under damped or critically damped?

(2 marks)

Solution: **(a)** for $t < 0$, the inductors are of short-circuit. The total resistance connected to the voltage source is 10 Ω and thus the current drawn from the source is 1 A, which will be equally distributed to the two parallel branches. Hence, $i_1 = i_2 = 0.5$ A.

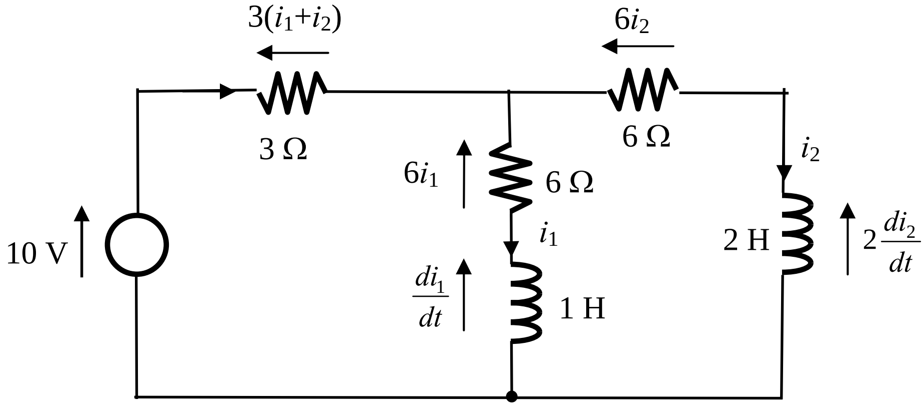


(b) Right after the switch is thrown to its final position, the inductor currents have to be continuous. Thus, $i_1 = i_2 = 0.5$ A, which implies the current passing the 3 Ω resistor is 1 A.



From the circuit above, it is clear that $v_1 = v_2 = 4$ V.

(c) Refer to the figure below.



Applying KVL to the left loop, we obtain

$$\frac{di_1}{dt} + 6i_1 + 3(i_1 + i_2) = 10 \quad \Rightarrow \quad \frac{di_1(t)}{dt} + 9i_1(t) + 3i_2(t) = 10 \quad \Rightarrow \quad 6i_2(t) = 20 - 2\frac{di_1(t)}{dt} - 18i_1(t)$$

$$\Rightarrow \quad \frac{d^2i_1(t)}{dt^2} + 9\frac{di_1(t)}{dt} + 3\frac{di_2(t)}{dt} = 0 \quad \Rightarrow \quad 2\frac{di_2(t)}{dt} = -\frac{2}{3}\frac{d^2i_1(t)}{dt^2} - 6\frac{di_1(t)}{dt}$$

Applying KVL to the right loop, we obtain

$$\frac{di_1(t)}{dt} + 6i_1(t) = 2\frac{di_2(t)}{dt} + 6i_2(t) = -\frac{2}{3}\frac{d^2i_1(t)}{dt^2} - 6\frac{di_1(t)}{dt} + 20 - 2\frac{di_1(t)}{dt} - 18i_1(t)$$

Thus, we have

$$\frac{2}{3}\frac{d^2i_1(t)}{dt^2} + 9\frac{di_1(t)}{dt} + 24i_1(t) = 20$$

(d) The characteristic polynomial is given by

$$\frac{2}{3}z^2 + 9z + 24 = 0$$

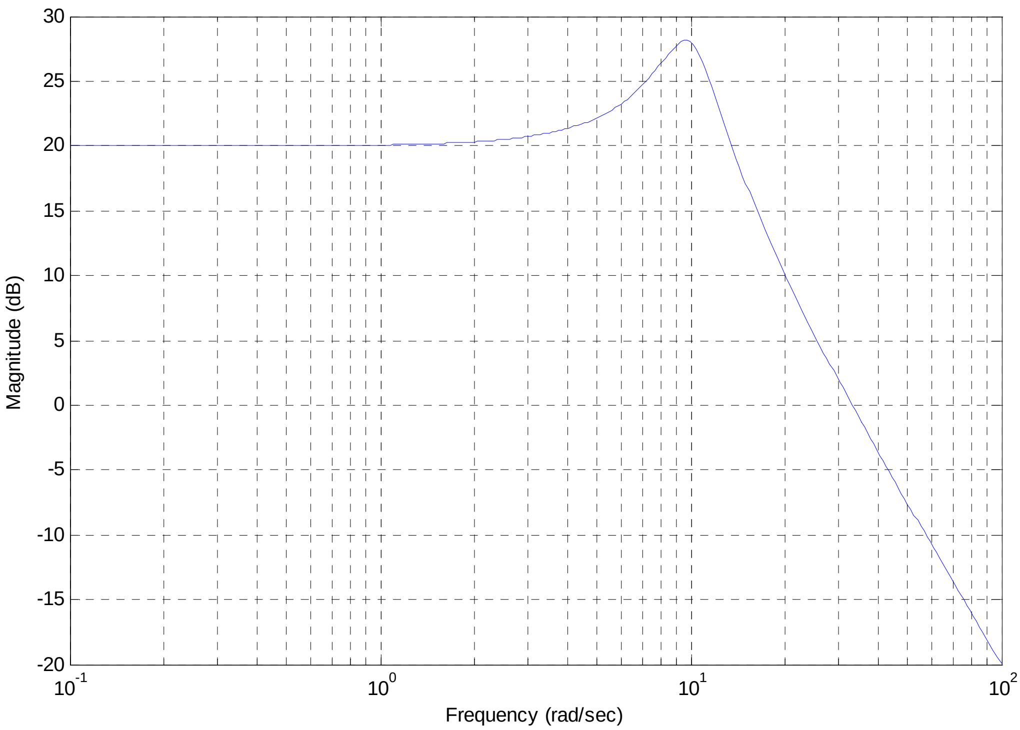
and its roots are $-9.8423, -3.6577$.

(e) The circuit is over damped as its characteristic polynomial has two distinct real roots.

Q.2. The magnitude response of a typical second order system characterized by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

is given in Fig. 2 below.



(a) Find the DC gain, K, the damping ratio, ζ, and the natural frequency, ω_n, of the given system.

(10 marks)

(b) Given an input signal, $u(t) = \cos 10t$, find its corresponding steady-state output, $y(t)$.

(5 marks)

(c) Find the overshoot, rise time, peak time and settling time of the unit step response of the system.

(5 marks)

(d) Sketch the unit step response.

(5 marks)

Solution:

(a) It is simple to observe from the magnitude response that the DC gain is 20 dB, i.e.,

$$K = 10.$$

The corner frequency, which is also the natural frequency, of the magnitude response is about 10 rad/sec, i.e.,

$$\omega_n = 10 \text{ rad/sec.}$$

The peak at the corner frequency is about 28 dB with 20 dB being produced by the DC gain. Thus, the peak without the contribution from the DC gain is 8 dB, which is corresponding to a damping ratio $\zeta = 0.2$. Thus, the transfer function is given by

$$H(s) = \frac{Y(s)}{U(s)} = \frac{10^3}{s^2 + 4s + 10^2}$$

(b) For the given input, we have $\omega = 10$ rad/sec. Its corresponding frequency response is given by

$$H(j\omega)\big|_{\omega=10} = \frac{10^3}{(j10)^2 + j40 + 10^2} = \frac{25}{j} = 25\angle -90^\circ$$

Thus, the corresponding steady-state output is given by

$$y(t) = 25\cos(10\,t - 90^\circ)$$

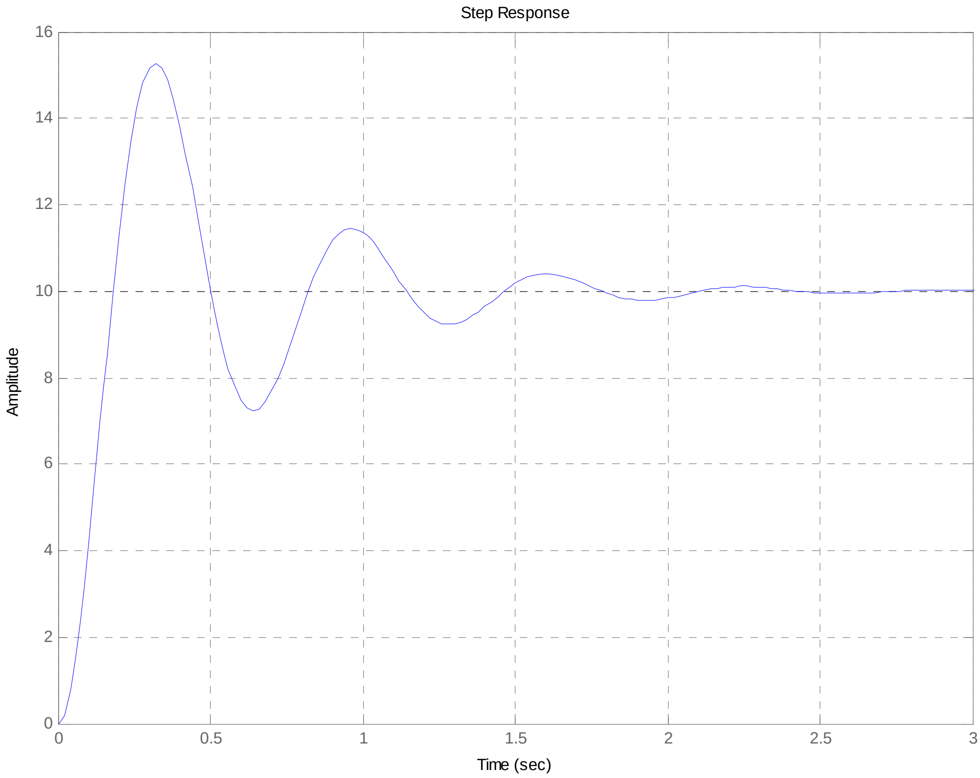
(c) Overshoot is given by $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} = e^{-3.14\times 0.2/\sqrt{1-0.2^2}} = 0.53 = 53\%$

Rise time $t_r = \dfrac{1.8}{\omega_n} = 0.18 \quad \text{sec}$

Peak time $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{3.14}{10\sqrt{1-0.2^2}} = 0.32 \quad \text{sec}$

Settling time $t_s = \dfrac{4}{\zeta\omega_n} = 2 \quad \text{sec}$

(d) Step response…


Step Response
Amplitude
Time (sec)
16
14
12
10
8
6
4
2
0
0
0.5
1
1.5
2
2.5
3