Q.1. An input-output relationship of a thermometer can be modeled by the differential equation below:

$$4\frac{dy(t)}{dt} + y(t) = 0.8u(t)$$

where $u(t)$ is the temperature of the environment in which the thermometer is placed, and $y(t)$ is the measured temperature.

The thermometer is inserted into a heat bath and the temperature reading is allowed to be stabilized before the temperature of the water in the heat bath is increased at a steady rate of 2°C/second. Assume that the heat bath temperature has been stabilized for a long time and at the instant $t = 0$ when the bath temperature starts to increase.

(a) Suppose the measured temperature is 40°C when t = 0, i.e. $y(0) = 40$°C. What is the temperature of the heat bath?

Solution: Since the heat bath temperature has been stabilized for a long time, thus for $t \le 0$

$$4\frac{dy(t)}{dt} + y(t) = y(t) = 0.8u(t) \quad \Rightarrow \quad u(t) = \frac{y(t)}{0.8} \quad \Rightarrow \quad u(0) = \frac{y(0)}{0.8} = \frac{40}{0.8} = 50°\text{C}$$ (5 marks)

(b) Obtain a mathematical expression to represent the actual temperature in the heat bath, $u(t)$.

Solution: $u(t) = u(0) + 2t = 50 + 2t$ (5 marks)

(c) Solve the differential equation to obtain the measured temperature in the time-domain, i.e., $y(t)$.

Solution: We look for two types of solutions, i.e., the transient response and the steady state response. Since $u(t) = u(0) + 2t = 50 + 2t$, we look for a steady state solution in the form of $y_{ss}(t) = k_1 + k_2 t$, which implies

$$4\frac{dy_{ss}(t)}{dt} + y_{ss}(t) = (4k_2 + k_1) + k_2 t = 0.8(50 + 2t) = 40 + 1.6t \quad \Rightarrow \quad k_2 = 1.6,\ k_1 = 33.6$$

For the transient response,

$$4\frac{dy(t)}{dt} + y(t) = 0 \quad \Rightarrow \quad 4z + 1 = 0 \quad \Rightarrow \quad z = -\frac{1}{4} = -0.25 \quad \Rightarrow \quad y_{tr}(t) = ke^{-0.25t}$$

The complete solution

$$y(t) = y_{tr}(t) + y_{ss}(t) = ke^{-0.25t} + 33.6 + 1.6t \quad \Rightarrow \quad y(0) = k + 33.6 = 40 \quad \Rightarrow \quad k = 6.4$$

Thus,

$$y(t) = 6.4e^{-0.25t} + 33.6 + 1.6t$$ (10 marks)

(d) Is the system (i) linear? (ii) time invariant? and (iii) BIBO stable?

Solution: The system is linear, time invariant and BIBO stable. (5 marks)

Q.2. The thermometer in Q.1 is apparently poorly designed. The manufacture has enhanced its design in a new model, which is governed by the following differential equation:

$$\frac{d^2y(t)}{dt^2}+11\frac{dy(t)}{dt}+10y(t)=10u(t)$$

where $u(t)$ is again the temperature of the environment in which the thermometer is placed, and $y(t)$ is the measured temperature.

(a) Find the transfer function of the thermometer system. Is the system stable?

Solution:

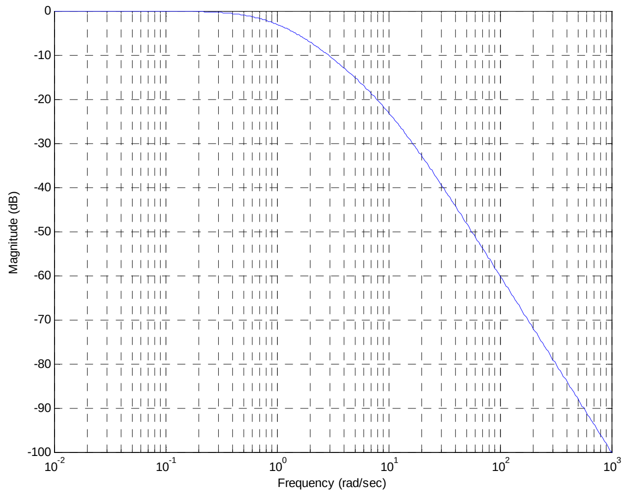
$$L\left\{\frac{d^2y(t)}{dt^2}+11\frac{dy(t)}{dt}+10y(t)\right\}=L\{10u(t)\} \quad\Rightarrow\quad s^2Y(s)+11sY(s)+10Y(s)=10U(s)$$

and thus, $H(s)=\frac{Y(s)}{U(s)}=\frac{10}{s^2+11s+10}$, which has two poles at $s_1=-1$ and $s_1=-10$, respectively. The system is stable. (5 marks)

(b) What is the DC gain of the thermometer system? If the thermometer is placed in an environment with a constant temperature, what is its steady state error of the measurement?

Solution: $H(0)\frac{10}{0^2+11\times 0+10}=1$. The steady state error for measuring constant temperature is 0. (5 marks)

(c) Sketch the frequency response (magnitude response only) of the system. (10 marks)



(d) Is the thermometer good enough to measure $u(t)=35\cos(100t)$? Why?

Solution: The thermometer is not good enough. The amplitude of the output is only around 0.035°C instead.

(5 marks)