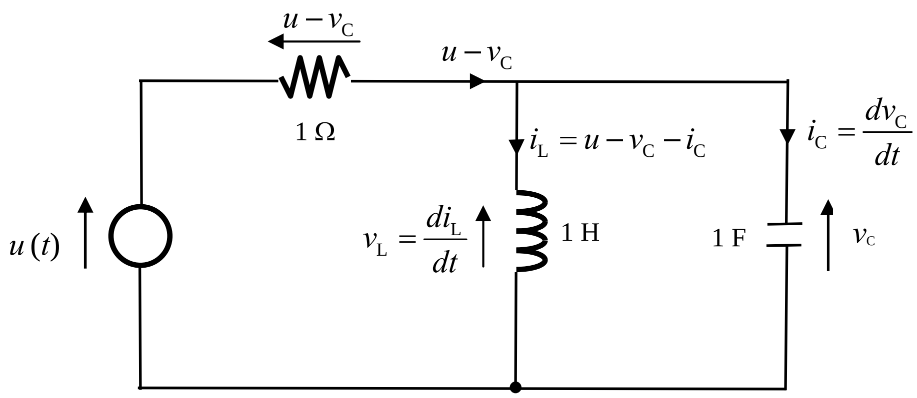
Q.1. In the electrical circuit given below, $u(t) = 10$ V, and both the capacitor and inductor are initially discharged.



(a) Show the differential equation governing the circuit in terms of v_C is given as

$$\frac{d^2 v_C(t)}{dt^2} + \frac{dv_C(t)}{dt} + v_C(t) = \frac{du(t)}{dt}$$

Give all the detailed derivations in obtaining the above differential equation.

(10 marks)

Solution: From all the currents and voltages labeled in the circuit above, we have

$$v_C = v_L = \frac{di_L}{dt} = \frac{d}{dt}(u - v_C - i_C) = \frac{d}{dt}\left(u - v_C - \frac{dv_C}{dt}\right) = \frac{du}{dt} - \frac{dv_C}{dt} - \frac{d^2 v_C}{dt^2}$$

$$\Downarrow$$

$$\frac{d^2 v_C}{dt^2} + \frac{dv_C}{dt} + v_C = \frac{du}{dt}$$

(b) Compute the roots of the characteristic polynomial of the circuit.

(5 marks)

Solution: The characteristic polynomial is given by

$$z^2 + z + 1 = 0 \quad \Rightarrow \quad z_1, z_2 = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} = -0.5 \pm j0.866$$

(c) Determine the steady-state values of i_L, i_C and v_C.

(5 marks)

Solution: $i_L(\infty)=10,\ i_C(\infty)=0,\ v_C(\infty)=0$

(d) Determine the complete solution of $v_C(t)$.

(5 marks)

Solution: The complete solution is given by

$$v_C(t)=v_{C,ss}+v_{C,tr}=0+k_1e^{z_1t}+k_2e^{z_2t}=k_1e^{(-0.5+j0.866)t}+k_2e^{(-0.5-j0.866)t}$$

Since the capacitor is initially discharged, we have

$$v_C(0)=k_1+k_2=0$$

Also,

$$\begin{aligned} i_L &= u-v_C-\frac{dv_C}{dt} \\ &=10-k_1e^{(-0.5+j0.866)t}-k_2e^{(-0.5-j0.866)t}-(-0.5+j0.866)k_1e^{(-0.5+j0.866)t}-(-0.5-j0.866)k_2e^{(-0.5-j0.866)t} \end{aligned}$$

The condition that the inductor is initially discharged implies

$$\begin{aligned} i_L(0) &=10-k_1-k_2-(-0.5+j0.866)k_1-(-0.5-j0.866)k_2 \\ &=10-j0.866(k_1-k_2)=0 \\ &\Downarrow \end{aligned}$$

$$k_2-k_1=j11.5473 \quad \Rightarrow \quad k_1=-j5.7737,\quad k_2=j5.7737$$

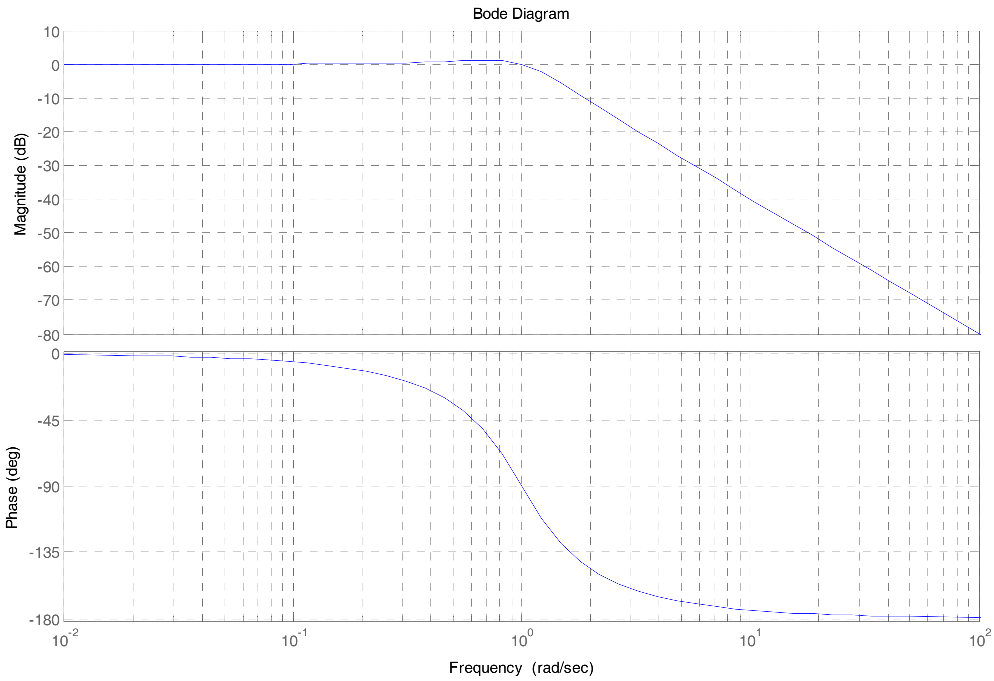
Hence,

$$v_C(t)=-j5.7737e^{(-0.5+j0.866)t}+j5.7737e^{(-0.5-j0.866)t}=11.5473e^{-0.5t}\sin 0.866t$$

Q.2. It can be shown that the electric system given in Q.1 has a transfer function from the inductor current i_L to the input voltage source u is given by

$$H(s) = \frac{I_L(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

and its Bode plot is given as in the figure below.



(a) Find the DC gain, K, the damping ratio, ζ, and the natural frequency, ω_n, of the given system.

(5 marks)

<u>Solution:</u> Observing that

$$H(s) = \frac{I_L(s)}{U(s)} = \frac{1}{s^2 + s + 1} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \Rightarrow \quad K = 1,\ \omega_n = 1,\ \zeta = 0.5$$

(b) Given an input, $u(t) = \cos t$, find its corresponding steady-state output, $i_L(t)$.

(5 marks)

Solution: $i_L(t) = \cos(t - 90^\circ)$

(c) Given an input, $u(t) = \cos 100\, t$, find its corresponding steady-state output, $i_L(t)$.

(5 marks)

Solution: $i_L(t) = 10^{-4}\cos(100t - 180^\circ)$

(d) Sketch the unit step response.

(10 marks)

Solution: