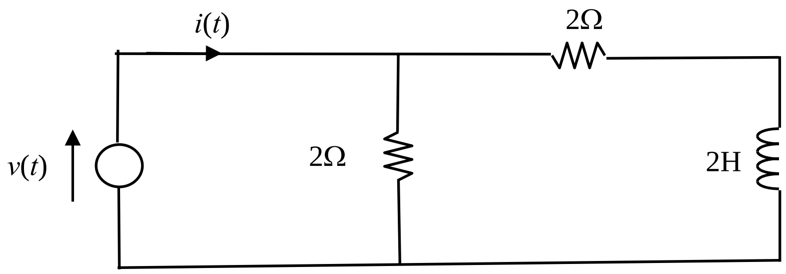
EE2010E Systems and Control Part 1 – Solutions to Tutorial Set 1

Q.1. In the following circuit (or electrical system), $v(t)$ is the system input and $i(t)$ is the system output.



(a) Derive a time-domain model for the circuit.

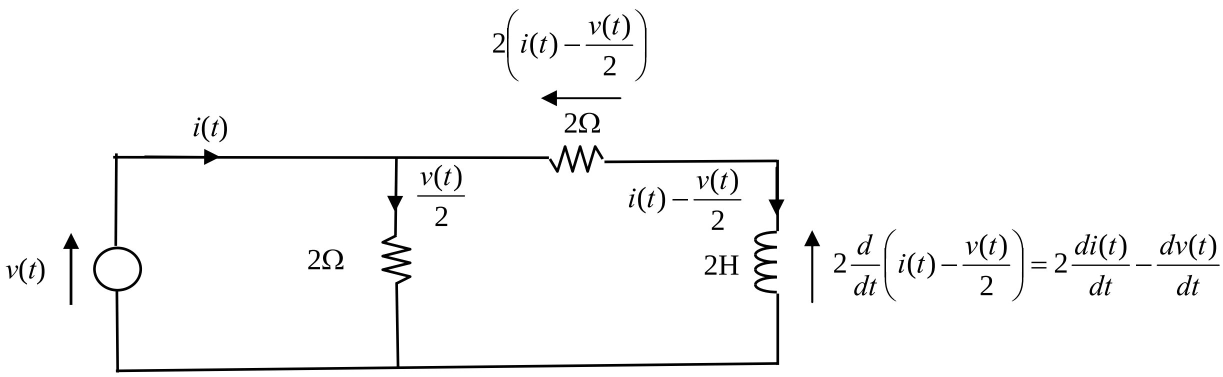
(b) Is the system is linear?

(c) Is the system is time invariant?

(d) Is that the system is causal?

(e) Is that the system is BIBO stable?

Solutions: **(a)** Refer to the voltages and currents marked in the figure below.



Applying KVL to the outer loop, we obtain the following equation:

$$2\frac{di(t)}{dt} - \frac{dv(t)}{dt} + 2i(t) - v(t) = v(t)$$

which gives a time-domain model of the circuit:

$$2\frac{di(t)}{dt} + 2i(t) = \frac{dv(t)}{dt} + 2v(t) \quad \Leftrightarrow \quad \frac{di(t)}{dt} + i(t) = 0.5\frac{dv(t)}{dt} + v(t)$$

(b) Let $i_1(t)$ be the output produced by $v_1(t)$ and $i_2(t)$ be the output produced by $v_2(t)$, i.e.

$$\frac{di_1(t)}{dt}+i_1(t)=0.5\frac{dv_1(t)}{dt}+v_1(t) \quad \& \quad \frac{di_2(t)}{dt}+i_2(t)=0.5\frac{dv_2(t)}{dt}+v_2(t)$$

We check if $i(t)=\alpha_1 i_1(t)+\alpha_2 i_2(t)$ is an output produced by $v(t)=\alpha_1 v_1(t)+\alpha_2 v_2(t)$. Observing that

$$\begin{aligned}\frac{di(t)}{dt}+i(t)&=\frac{d}{dt}\big(\alpha_1 i_1(t)+\alpha_2 i_2(t)\big)+\big(\alpha_1 i_1(t)+\alpha_2 i_2(t)\big)\\&=\alpha_1\left[\frac{di_1(t)}{dt}+i_1(t)\right]+\alpha_2\left[\frac{di_2(t)}{dt}+i_2(t)\right]\\&=\alpha_1\left[0.5\frac{dv_1(t)}{dt}+v_1(t)\right]+\alpha_2\left[0.5\frac{dv_2(t)}{dt}+v_2(t)\right]\\&=0.5\frac{d}{dt}\big(\alpha_1 v_1(t)+\alpha_2 v_2(t)\big)+\big(\alpha_1 v_1(t)+\alpha_2 v_2(t)\big)\\&=0.5\frac{dv(t)}{dt}+v(t)\end{aligned}$$

$i(t)=\alpha_1 i_1(t)+\alpha_2 i_2(t)$ is indeed an output produced by $v(t)=\alpha_1 v_1(t)+\alpha_2 v_2(t)$. By definition, the circuit (or the system) is linear.

(c) To see if the system is time invariant, let us do it by following the procedure given in the notes.

Step One: Suppose $i_1(t)$ is a solution corresponding to a voltage input $v_1(t)$.

$$\frac{di_1(t)}{dt}+i_1(t)=0.5\frac{dv_1(t)}{dt}+v_1(t) \quad\Rightarrow\quad \frac{di_1(t-t_0)}{d(t-t_0)}+i_1(t-t_0)=0.5\frac{dv_1(t-t_0)}{d(t-t_0)}+v_1(t-t_0)$$

Step Two: Let $v_2(t)=v_1(t-t_0)$. Verify if $i_2(t)=i_1(t-t_0)$ is a solution to the circuit (system):

$$\begin{aligned}\frac{di_2(t)}{dt}+i_2(t)&=\frac{di_1(t-t_0)}{dt}+i_1(t-t_0)=\frac{di_1(t-t_0)}{d(t-t_0)}+i_1(t-t_0)\\&=0.5\frac{dv_1(t-t_0)}{d(t-t_0)}+v_1(t-t_0)\\&=0.5\frac{dv_1(t-t_0)}{dt}+v_1(t-t_0)=0.5\frac{dv_2(t)}{dt}+v_2(t)\end{aligned}$$

which shows that $i_2(t)$ is indeed a solution corresponding to $v_2(t)$. By definition, the system is time-invariant.

Exercise: Show that in general, the following system is linear and time-invariant:

$$a_n\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\cdots+a_1\frac{dy(t)}{dt}+y(t)=b_m\frac{d^m u(t)}{dt^m}+b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}}+\cdots+b_1\frac{du(t)}{dt}+u(t)$$

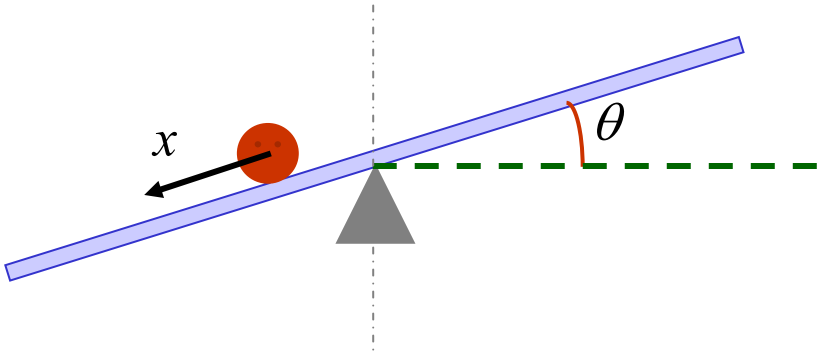
(d) Yes. The system is causal as the output at time t_0 is depended only on the input for $t \le t_0$.

(e) Yes. The circuit is BIBO stable, which can be judged either from the physical properties of the circuit or from mathematical derivations.

Physically, for any bounded voltage source, $v(t)$, the resulting current, $i(t)$, is always bounded. Why?

Exercise: Let $v(t)$ be a bounded DC source, prove mathematically that $i(t)$ is bounded.

Q.2. Consider a ball and beam balancing mechanical system below. Let θ be the system input and let x, the displacement of the ball, be the system output. Assume that there is no friction on the surfaces.



(a) Derive a time-domain model for the mechanical system.

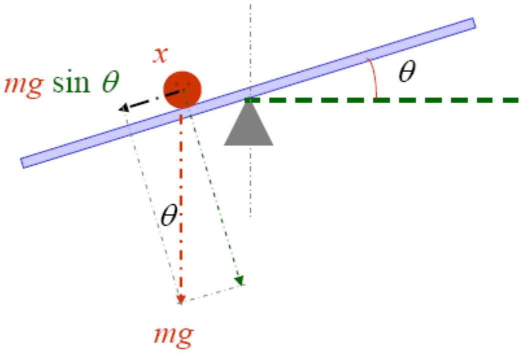
(b) Is the system is linear?

(c) Is the system is time invariant?

(d) Is that the system is causal?

(e) Is that the system is BIBO stable?

<u>Solution:</u> (a) Since there is no friction on the surfaces, the only force acts on the system is the weight of the ball, i.e.



By Newton's law of motion, we have

$$F = ma \quad \Rightarrow \quad mg\sin\theta = ma = m\ddot{x} \quad \Rightarrow \quad \ddot{x} = g\sin\theta$$

where g is the gravity constant, i.e., g = 9.8. Thus, the time-domain model of the system is

$$\ddot{x} = 9.8\sin\theta \quad \Leftrightarrow \quad \frac{d^2x(t)}{dt^2} = 9.8\sin\theta(t)$$

(b) Assume that the ball is initially stationary, i.e. $x(0) = 0$ and $\dot{x}(0) = 0$ **.** Let θ_1=10° and let $x_1(t)$ be the corresponding solution, i.e.,

$$\frac{d^2 x_1(t)}{dt^2} = 9.8 \sin 10° = 1.7018 \quad \Rightarrow \quad x_1(t) = 0.8509 t^2$$

Let $\theta = \alpha\, \theta_1 = 3 \times 10° = 30°$. However, it can be verified that the corresponding solution $x(t) \neq \alpha\, x_1(t)$**,** i.e.,

$$\frac{d^2 x(t)}{dt^2} = 9.8 \sin 30° = 4.9 \quad \Rightarrow \quad x(t) = 2.45 t^2 \neq 3 x_1(t) = 2.5527 t^2$$

Thus, the system is nonlinear.

(c) The system is time-invariant. This can be verified by the following steps.

Step One: Suppose $x_1(t)$ is a solution corresponding to $\theta_1(t)$.

$$\frac{d^2 x_1(t)}{dt^2} = 9.8 \sin \theta_1(t) \quad \Rightarrow \quad \frac{d^2 x_1(t-t_0)}{[d(t-t_0)]^2} = 9.8 \sin \theta_1(t-t_0)$$

Step Two: Let $\theta_2(t) = \theta_1(t-t_0)$. Verify if $x_2(t) = x_1(t-t_0)$ is a solution to the system:

$$\frac{d^2 x_2(t)}{dt^2} = \frac{d^2 x_1(t-t_0)}{dt^2} = \frac{d^2 x_1(t-t_0)}{[d(t-t_0)]^2} = 9.8 \sin \theta_1(t-t_0) = 9.8 \sin \theta_2(t)$$

which shows that $x_2(t)$ is indeed a solution corresponding to $\theta_2(t)$. By definition, the system is time-invariant.

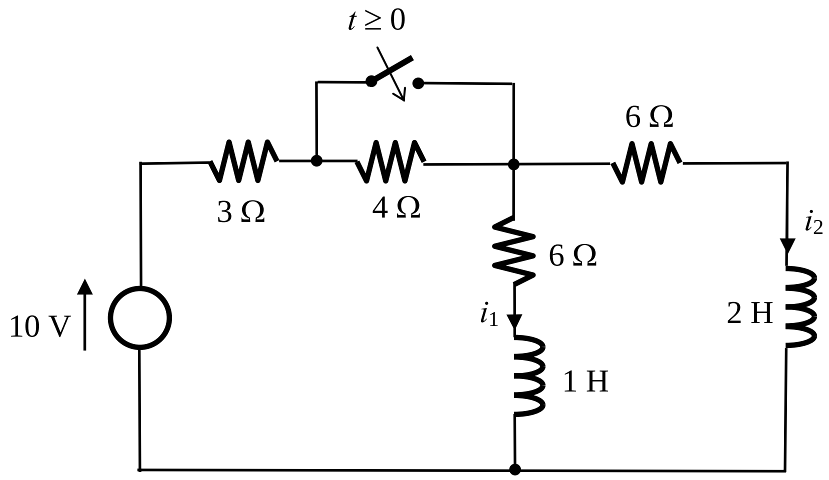
(d) It is obvious that the system is causal.

(e) The system is not BIBO stable. We show this by a specific example. Let the ball be initially stationary, i.e. $x(0) = 0$ and $\dot{x}(0) = 0$ **,** and let $\theta = 1°$, which is bounded.

$$\frac{d^2 x(t)}{dt^2} = 9.8 \sin 1° = 0.171 \quad \Rightarrow \quad x(t) = 0.0855 t^2 \to \infty \quad \text{as } t \to \infty$$

Clearly, $x(t)$ is unbounded. Thus, the system is BIBO unstable.

Q.3. In the electrical circuit given below, the switch has been in the position shown for a long time and is thrown to the other position for time $t \geq 0$.



(a) Determine the currents for both inductors for $t < 0$.

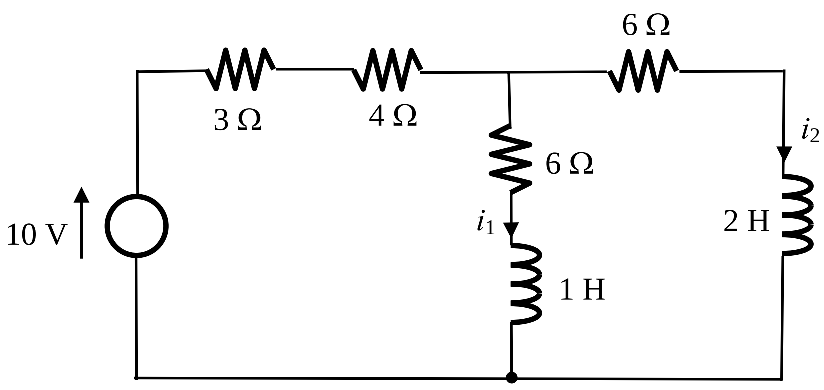
(b) Determine the currents and voltages for both inductors just right after the switch is closed.

(c) Derive the differential equation governing the circuit in terms of i_1.

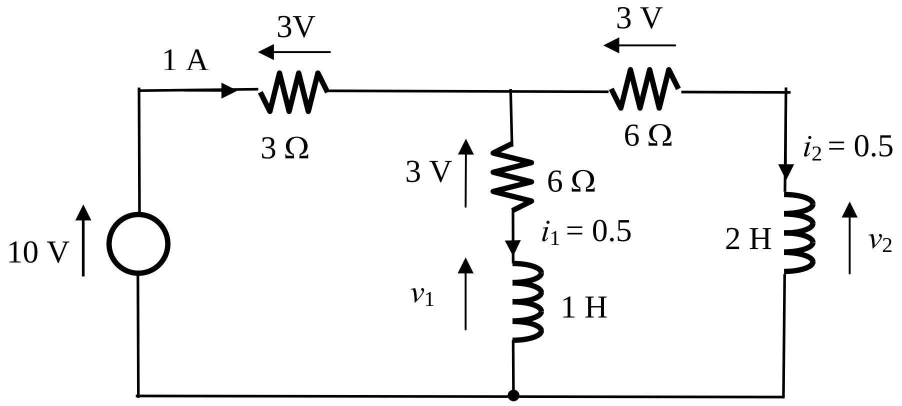
(d) Compute the roots of its characteristic polynomial.

(e) Is the circuit over damped, under damped or critically damped?

Solution: **(a)** for $t < 0$, the inductors are of short-circuit. The total resistance connected to the voltage source is 10 Ω and thus the current drawn from the source is 1 A, which will be equally distributed to the two parallel branches. Hence, $i_1 = i_2 = 0.5$ A.

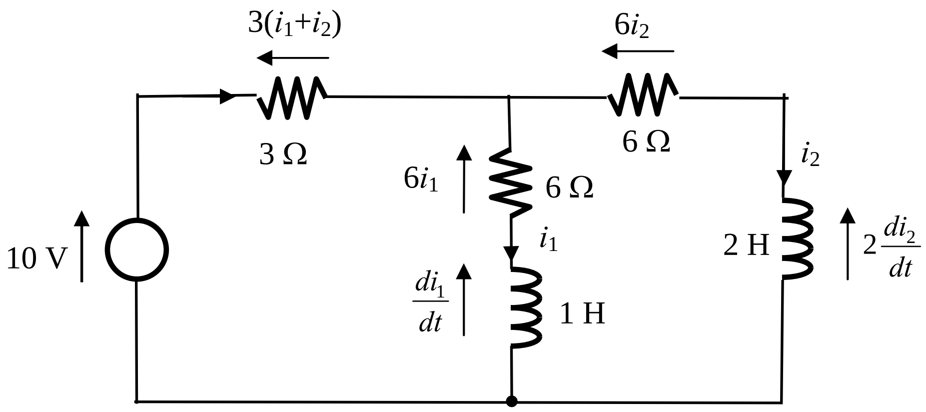


(b) Right after the switch is thrown to its final position, the inductor currents have to be continuous. Thus, $i_1 = i_2 = 0.5$ A, which implies the current passing the 3 Ω resistor is 1 A.



From the circuit above, it is clear that $v_1 = v_2 = 4$ V.

(c) Refer to the figure below.



Applying KVL to the left loop, we obtain

$$\frac{di_1}{dt} + 6i_1 + 3(i_1 + i_2) = 10 \quad \Rightarrow \quad \frac{di_1(t)}{dt} + 9i_1(t) + 3i_2(t) = 10 \quad \Rightarrow \quad 6i_2(t) = 20 - 2\frac{di_1(t)}{dt} - 18i_1(t)$$

$$\Rightarrow \quad \frac{d^2 i_1(t)}{dt^2} + 9\frac{di_1(t)}{dt} + 3\frac{di_2(t)}{dt} = 0 \quad \Rightarrow \quad 2\frac{di_2(t)}{dt} = -\frac{2}{3}\frac{d^2 i_1(t)}{dt^2} - 6\frac{di_1(t)}{dt}$$

Applying KVL to the right loop, we obtain

$$\frac{di_1(t)}{dt} + 6i_1(t) = 2\frac{di_2(t)}{dt} + 6i_2(t) = -\frac{2}{3}\frac{d^2 i_1(t)}{dt^2} - 6\frac{di_1(t)}{dt} + 20 - 2\frac{di_1(t)}{dt} - 18i_1(t)$$

Thus, we have

$$\frac{2}{3}\frac{d^2 i_1(t)}{dt^2} + 9\frac{di_1(t)}{dt} + 24i_1(t) = 20$$

(d) The characteristic polynomial is given by

$$\frac{2}{3}z^2 + 9z + 24 = 0$$

and its roots are $-9.8423, -3.6577$.

(e) The circuit is over damped as its characteristic polynomial has two distinct real roots.

Q.4. An input-output relationship of a thermometer can be modeled by the following differential equation:

$$5\frac{dy(t)}{dt}+y(t)=0.99u(t)$$

where $u(t)$ is the temperature of the environment in which the thermometer is placed, and $y(t)$ is the measured temperature.

The thermometer is inserted into a heat bath and the temperature reading is allowed to be stabilized before the temperature of the water in the heat bath is increased at a steady rate of 1°C/second. Assume that $t = 0$ at the instant when the hot bath temperature starts to increase.

(a) Suppose the measured temperature is 24.75°C when t = 0, i.e. $y(0) = 24.75$°C. What is the temperature of the heat bath?

(b) Write a mathematical expression to represent the temperature in the heat bath, $u(t)$. Then solve the differential equation to obtain the time-domain expression of the measured temperature, $y(t)$.

Solution:

(a) The input-output relationship of the thermometer is

$$5\frac{\mathrm{d}y(t)}{\mathrm{d}t}+y(t)=0.99u(t)$$

When the temperature reading stabilises, $\frac{\mathrm{d}y(t)}{\mathrm{d}t}=0$ so the differential equation reduces to

$$y(t)=0.99u(t)$$

Given that $y = 24.75$, the temperature of the heat bath is

$$u=\frac{y}{0.99}=25^{o}\mathrm{C}.$$

(b) Initial heat bath temperature is 25°C and it increases at a steady rate of 1°C/second.

$$u(t) = [25 + t]$$

Substituting u(t) into the differential equation, the time-domain expression for the measured temperature can be found by solving

$$5\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = 0.99u(t) \qquad \text{where } y(0) = 24.75$$

We are looking for both the steady-state solution and the transient response. For the steady-state, we test a solution

$$y_{ss}(t) = k_1 + k_2 t$$

Substituting it into the differential equation, we have

$$5\frac{dy_{ss}(t)}{dt} + y_{ss}(t) = 5k_2 + k_1 + k_2 t = 0.99u(t) = 0.99(25 + t)$$

Thus, we have $k_1 = 19.8$ and $k_2 = 0.99$. Hence, $y_{ss}(t) = 19.8 + 0.99t$.

The characteristic polynomial of the differential equation is given by

$$5z + 1 = 0 \quad \Rightarrow \quad z = -0.2$$

Thus, the transient response is given by $y_{tr}(t) = ke^{-0.2t}$ and the complete response is

$$y(t) = y_{ss}(t) + y_{tr}(t) = 19.8 + 0.99t + ke^{-0.2t}$$

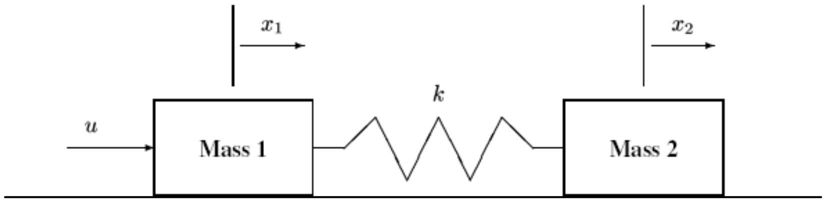
The initial condition

$$y(0) = 19.8 + k = 24.75 \quad \Rightarrow \quad k = 4.95$$

The final solution is then given by

$$y(t) = 19.8 + 0.99t + 4.95e^{-0.2t}$$

Q.5. Consider a two-mass-spring flexible mechanical system given below.



In the system, $u(t)$ is the input force, k = 1 is the spring constant, x_1 and x_2 are, respectively, the displacements of Mass 1 and Mass 2, which have masses of $m_1 = m_2 = 1$. Assume that there is no friction on the surfaces.

(a) Drive a differential equation of the mechanical system in terms of the displacement of Mass 2, i.e. x_2.

(b) Assuming that $u(t) = 1$ and the masses are initially stationary, show that $x_2(t) = 0.25t^2$ is a solution to the differential equation obtained in (a).

(c) Is the system BIBO stable?

<u>Solution:</u> (a) Applying Newton's Law of motion to Mass 1 and Mass 2, we obtain

$$m_1\ddot{x}_1 = k(x_2 - x_1) + u \quad \Rightarrow \quad m_1\ddot{x}_1 + kx_1 - kx_2 = u$$
$$m_2\ddot{x}_2 = k(x_1 - x_2)$$

The second equation implies

$$kx_1 = m_2\ddot{x}_2 + kx_2 \quad \& \quad k\ddot{x}_1 = m_2\frac{d^4x_2}{dt^4} + k\ddot{x}_2 \quad \Rightarrow \quad m_1\ddot{x}_1 = \frac{m_1m_2}{k}\frac{d^4x_2}{dt^4} + m_1\ddot{x}_2$$

Substituting these into the first equation, we obtain

$$m_1\ddot{x}_1 + kx_1 - kx_2 = u \quad \Rightarrow \quad \frac{m_1m_2}{k}\frac{d^4x_2}{dt^4} + m_1\ddot{x}_2 + m_2\ddot{x}_2 + kx_2 - kx_2 = u$$

or

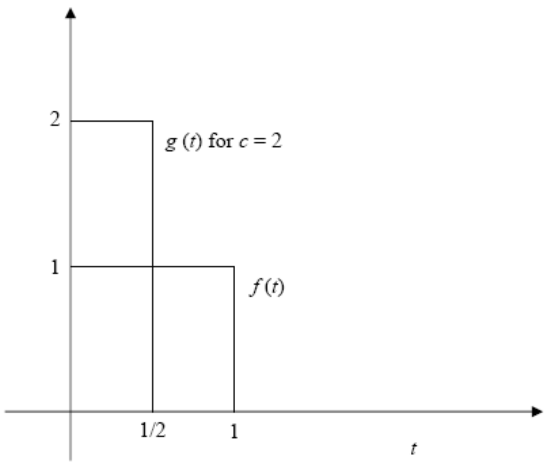
$$\frac{m_1m_2}{k}\frac{d^4x_2}{dt^4} + (m_1 + m_2)\ddot{x}_2 = u \quad \Rightarrow \quad \frac{d^4x_2}{dt^4} + 2\frac{d^2x_2}{dt^2} = u$$

(b) It is simple to verify that $\frac{d^4(0.25t^2)}{dt^4} + 2\frac{d^2(0.25t^2)}{dt^2} = 1 = u.$

(c) Obviously, the system is not BIBO stable.

EE2010E Systems and Control Part 1 – Solutions to Tutorial Set 2

Q.1. Consider the square pulse $f(t)$ show in figure below. If we compress the pulse by a factor $c > 1$ and at the same time amplify its amplitude by the same factor c, we get a new function $g(t)$ as shown in the figure ($c = 2$ for the given figure).



(a) Find the Laplace transform of the function $g(t)$ from the transform of $f(t)$.

(b) Comment on what happens if c gets very large.

Solution:

(a). $f(t) = u(t) - u(t-1)$. Its Laplace transform is

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s}.$$

$g(t) = cf(ct)$. Its Laplace transform (by time-frequency scaling) is

$$G(s) = c\left[\frac{1}{c}\left(\frac{1}{s/c} - \frac{e^{-s/c}}{s/c}\right)\right] = \frac{c}{s}\left(1 - e^{-s/c}\right).$$

Thus for a time-compression factor $c = 2$,

$$G(s) = \frac{2}{s}\left(1 - e^{-s/2}\right).$$

(b). As c gets larger and larger, $g(t)$ approaches the unit impulse function $\delta(t)$ [Its area is always 1 for any c, and $g(t)$ goes to zero for any non-zero t].

To evaluate the transform $G(s)$ as c gets very large, we may apply the Well-known Taylor series expansion of the exponential function,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots ,$$

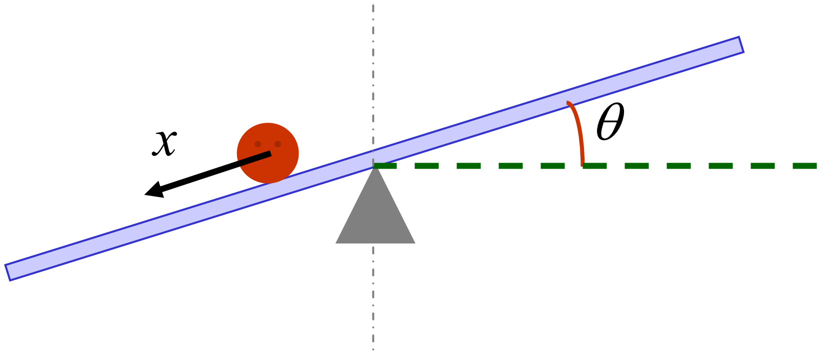
with $x = -s/c$, we get

$$G(s) = \frac{c}{s}\left[1 - \left(1 - \frac{s}{c} + \frac{s^2}{2!c^2} - \frac{s^3}{3!c^3} + \ldots\right)\right] = 1 - \frac{s}{2!c} + \frac{s^2}{3!c^2} + \ldots$$

As c gets very large, $G(s) \to 1$.

This is consistent with the transform of the function that $g(t)$ is approaching, since we know that $L[\delta(t)] = 1$.

Q.2. Consider the ball and beam balancing mechanical system again as in Tutorial Set 1. Let θ be the system input and let x, the displacement of the ball, be the system output. Assume that θ is changing in a very small range, i.e. $\sin\theta \approx \theta$.



(a) Find the transfer function of the system from the input θ to the output x.

(b) Find the unit impulse response of the system.

(c) Find the unit step response of the system.

Solution: **(a)** It was derived in Tutorial Set 1 that

$$\frac{d^2x(t)}{dt^2} = 9.8\sin\theta(t) \approx 9.8\theta$$

Thus, we have

$$s^2X(s) = 9.8\theta(s) \quad \Rightarrow \quad H(s) = \frac{X(s)}{\theta(s)} = \frac{9.8}{s^2}$$

(b) For the unit impulse input, we have

$$X(s) = \frac{9.8}{s^2}\theta(s) = \frac{9.8}{s^2} \quad \Rightarrow \quad x(t) = L^{-1}\left\{\frac{9.8}{s^2}\right\} = 9.8t$$

(c) For the unit step input, we have

$$X(s) = \frac{9.8}{s^2}\theta(s) = \frac{9.8}{s^3} \quad \Rightarrow \quad x(t) = L^{-1}\left\{\frac{9.8}{s^3}\right\} = 4.9t^2$$

Q.3. Use Laplace transform to solve the response $y(t)$ in the following integrodifferential equation:

$$\frac{dy(t)}{dt} + 5y(t) + 6\int_0^t y(\tau)d\tau = u(t), \quad y(0) = 2$$

Solution:

Taking the Laplace transform of each term, we get

$$[sY(s) - y(0^-)] + 5Y(s) + \frac{6}{s}Y(s) = \frac{1}{s}$$

Substituting $y(0)=2$ and multiplying throughout by s, we get

$$Y(s)(s^2 + 5s + 6) = 1 + 2s$$

Or

$$Y(s) = \frac{2s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

where

$$A = (s+2)Y(s)\Big|_{s=-2} = \left.\frac{2s+1}{s+3}\right|_{s=-2} = -3$$

$$B = (s+3)Y(s)\Big|_{s=-3} = \left.\frac{2s+1}{s+2}\right|_{s=-3} = 5$$

Thus,

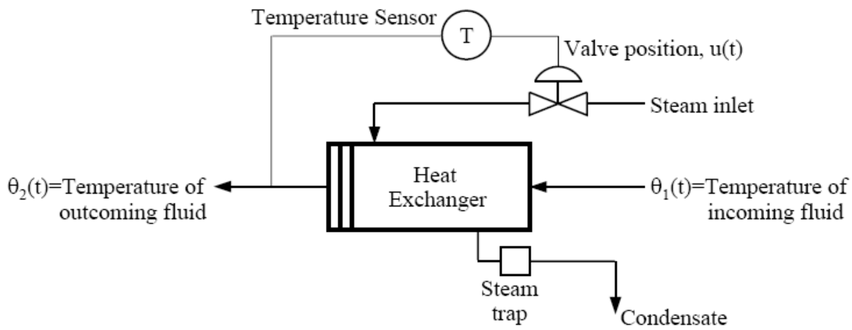
$$Y(s) = \frac{-3}{s+2} + \frac{5}{s+3}$$

Its inverse transform is

$$y(t) = (-3e^{-2t} + 5e^{-3t})$$

Q.4. Figure below shows a heat exchanger (a device for transferring heat from one fluid to another, where the fluids are separated by a solid wall so that they never mix). The temperature of the outgoing fluid, $\theta_2(t)$, needs to be maintained at a desired value, $\theta_r(t)$. Factors which influence the exit temperature are:

- The valve position, $u(t)$, which adjusts the flow of steam into the system.
- unmeasurable disturbances in the temperature of the incoming fluid stream, $\theta_1(t)$.



The dynamic behavior of the heat exchanger may be modeled by the following equation:

$$\theta_2(s) = \frac{2}{(s+1)^2}U(s) + \frac{1}{s+1}\theta_1(s)$$

Let the valve position $u(t) = 2\,[\theta_r(t) - \theta_2(t)]$, i.e. it is proportional to the error of the desired value and the actual outgoing temperature.

(a) If $\theta_r(t)$ is a unit step function and $\theta_1(t) = 0$, determine the transfer function $\theta_2(s)/\,\theta_r(s)$ and then use it to calculate $\theta_2(t)$. Identify the transient and steady-state components in the step response.

(b) Given that $\theta_1(t)$ is a unit step function and $\theta_r(t) = 0$, find the transfer function $\theta_2(s)/\,\theta_1(s)$ and $\theta_2(t)$.

(c) Use superposition to obtain $\theta_2(t)$ given that both $\theta_r(t)$ and $\theta_1(t)$ are unit step functions. Find $\theta_2(\infty)$.

(d) Use the final value theorem instead to find $\theta_2(\infty)$ and compare it with the answer obtained in Part (c).

Solution:

(a) Assume that $\theta_1 = 0$. Hence, the temperature of the outgoing liquid is governed by

$$\begin{aligned}
\theta_2(s) &= \frac{2\times 2}{(s+1)^2}(\theta_r - \theta_2) \\
\left[1 + \frac{4}{(s+1)^2}\right]\theta_2(s) &= \frac{4}{(s+1)^2}\theta_r(s) \\
\text{Transfer function, } \frac{\theta_2(s)}{\theta_r(s)} &= \frac{4}{(s+1)^2 + 4} \\
&= \frac{4}{s^2 + 2s + 1 + 4} \\
&= \frac{4}{s^2 + 2s + 5}
\end{aligned}$$

When $\theta_r(t)$ is a unit step function,

$$\begin{aligned}
\theta_2(s) &= \frac{\theta_2(s)}{\theta_r(s)} \times \theta_r(s) \\
&= \frac{4}{s\,(s^2 + 2s + 5)} \\
&= \frac{0.8}{s} + \frac{-0.8s - 1.6}{s^2 + 2s + 5} \\
&= \frac{0.8}{s} + \frac{-0.8(s+1) - 0.8}{(s+1)^2 + 4} \\
&= \frac{0.8}{s} - 0.8\frac{s+1}{(s+1)^2 + 4} - 0.4\frac{2}{(s+1)^2 + 4} \\
\theta_2(t) &= 0.8 - 0.8e^{-t}\cos 2t - 0.4e^{-t}\sin 2t
\end{aligned}$$

Transient component of solution is $\theta_{2,tr} = -0.8e^{-t}\cos 2t - 0.4e^{-t}\sin 2t$

Steady-state component is $\theta_{2,ss} = 0.8$

(b) For $\theta_r = 0$, The s-domain expression of the temperature of the fluid leaving the heat exchanger is

$$\begin{aligned}
\theta_2(s) &= \frac{1}{s+1}\theta_1(s) + \frac{4}{(s+1)^2}(0-\theta_2) \\
\left[1+\frac{4}{(s+1)^2}\right]\theta_2(s) &= \frac{1}{s+1}\theta_1(s) \\
\text{Transfer function, } \frac{\theta_2(s)}{\theta_1(s)} &= \frac{1}{s+1} \times \frac{1}{1+4/(s+1)^2} \\
&= \frac{s+1}{s^2+2s+5}
\end{aligned}$$

When $\theta_1(t)$ is a unit step function,

$$\begin{aligned}
\theta_2(s) &= \frac{\theta_2(s)}{\theta_1(s)} \times \theta_1(s) \\
&= \frac{s+1}{s(s^2+2s+5)} \\
&= \frac{0.2}{s} + \frac{-0.2s+0.6}{s^2+2s+5} \\
&= \frac{0.2}{s} + \frac{-0.2(s+1)+0.8}{(s+1)^2+4} \\
\theta_2(t) &= 0.2 - 0.2e^{-t}\cos 2t + 0.4e^{-t}\sin 2t
\end{aligned}$$

(c) Using the principle of superposition, the temperature of the outgoing fluid when $\theta_r(t)$ and $\theta_1(t)$ are both unit step functions,

$$\begin{aligned}
\theta_2(t) &= \theta_2(t)|_{\theta_r(t)=\text{unit step},\theta_1(t)=0} + \theta_2(t)|_{\theta_r(t)=0,\theta_1(t)=\text{unit step}} \\
&= 0.8 - 0.8e^{-t}\cos 2t - 0.4e^{-t}\sin 2t \\
&\quad +0.2 - 0.2e^{-t}\cos 2t + 0.4e^{-t}\sin 2t \\
&= 1 - e^{-t}\cos 2t
\end{aligned}$$

Steady-state value of the temperature of the outgoing fluid is $\lim_{t\to\infty}\theta_2(t) = 1$

(d) The final theorem states that $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$. Hence, the steady-state temperature of the fluid exiting the heat exchanger is

$$\begin{aligned}
\lim_{t\to\infty}\theta_2(t) &= \lim_{s\to 0} s\theta_2(s) \\
&= \lim_{s\to 0} s\left[\frac{\theta_2(s)}{\theta_r(s)}\theta_r(s) + \frac{\theta_2(s)}{\theta_1(s)}\theta_1(s)\right] \\
&= \lim_{s\to 0} s\left[\frac{4}{s^2+2s+5}\,\frac{1}{s} + \frac{s+1}{s^2+2s+5}\,\frac{1}{s}\right] \\
&= 0.8 + 0.2 = 1
\end{aligned}$$

Q.5. Consider the first order system

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{\tau s + 1}$$

(a) Find the step response, $y_{\text{step}}(t)$.

(b) Find the impulse response, $y_{\text{impulse}}(t)$.

(c) Verify that

$$\dot{y}_{\text{step}}(t) = y_{\text{impulse}}(t) \qquad \text{and} \qquad \int_0^t y_{\text{impulse}}(\tau)d\tau = y_{\text{step}}(t)$$

Solution:

(a) Step response is the output of the system, $G(s) = \frac{1}{\tau s+1}$ when the input is a step function i.e. $U(s) = \frac{1}{s}$.

$$\begin{aligned} Y_{step}(s) &= G(s)U(s) \\ &= \frac{1}{s(\tau s + 1)} \\ &= \frac{1}{s} - \frac{\tau}{\tau s + 1} \\ &= \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}} \\ y_{step}(t) &= \mathcal{L}^{-1}\{Y_{step}(s)\} \\ &= 1 - e^{-\frac{t}{\tau}} \end{aligned}$$

(b) Impulse response is the output of the system when the input is an impulse function i.e. $U(s) = 1$

$$\begin{aligned} Y_{impulse}(s) &= G(s)U(s) \\ &= \frac{1}{\tau s + 1} \qquad \because U(s) = 1 \\ &= \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}} \\ y(t) &= \mathcal{L}^{-1}\{Y_{impulse}(s)\} \\ &= \frac{1}{\tau} e^{-\frac{t}{\tau}} \end{aligned}$$

(c) Differentiating the step response gives

$$\begin{aligned} \frac{\mathrm{d}y_{step}(t)}{\mathrm{d}t} &= \frac{1}{\tau} e^{-\frac{t}{\tau}} \\ &= y_{impulse}(t) \end{aligned}$$

Integrating the impulse response gives

$$\begin{aligned} \int_0^t y_{impulse}(x)\, dx &= -\, e^{-\frac{x}{\tau}} \Big|_0^t \\ &= 1 - e^{-\frac{t}{\tau}} \\ &= y_{step}(t) \end{aligned}$$

EE2010E Systems and Control Part 1 – Solutions to Tutorial Set 3

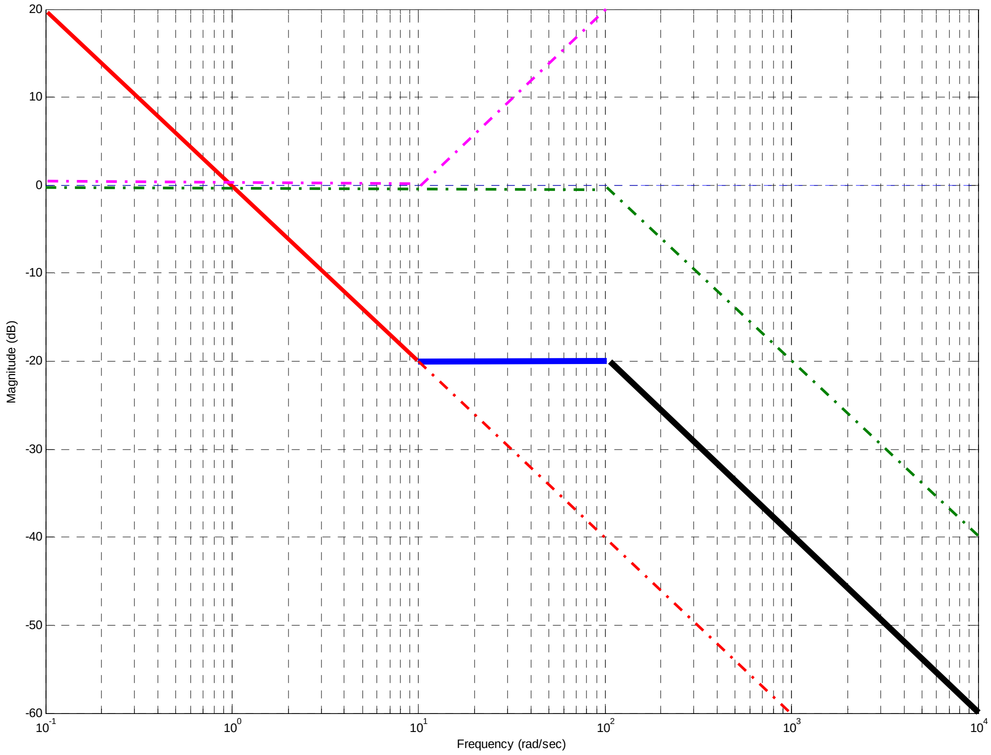
Q.1. Obtain the Bode plots for the following transfer function:

$$G(j\omega)=\frac{Y(j\omega)}{U(j\omega)}=\frac{10(j\omega+10)}{j\omega(j\omega+100)}$$

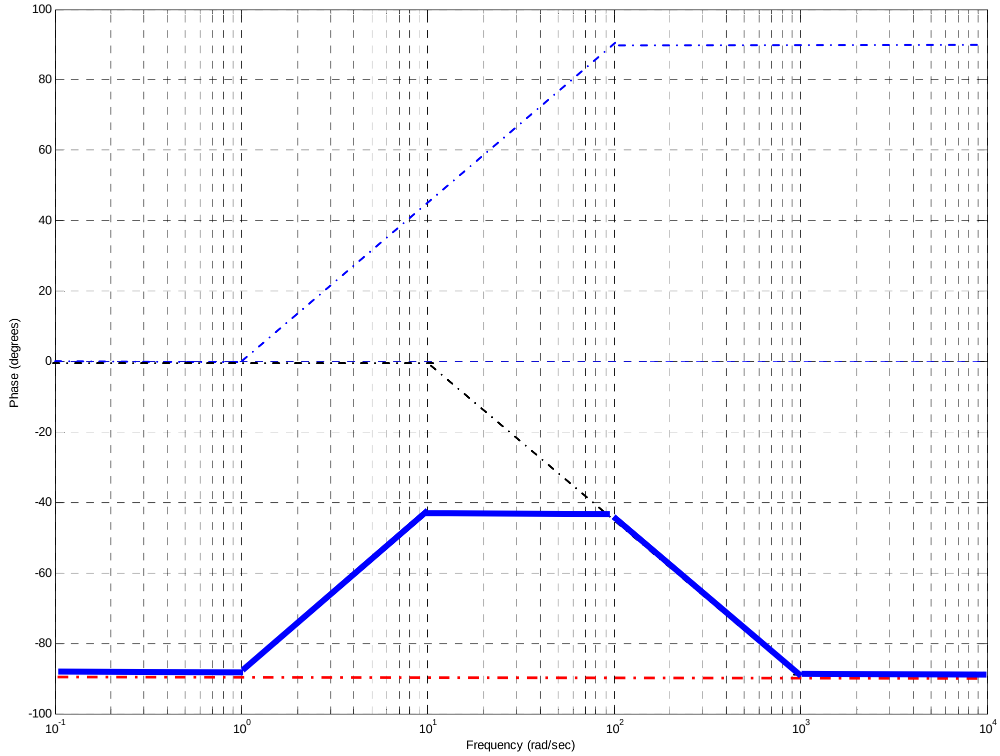
Given $u(t) = 5\cos(30t + 30°)$, find the corresponding output $y(t)$ using the Bode plots obtained above.

Solution:

$$G(j\omega)=\frac{Y(j\omega)}{U(j\omega)}=\frac{10(j\omega+10)}{j\omega(j\omega+100)}=\frac{10\times10(1+j\omega/10)}{j\omega100(1+j\omega/100)}=\frac{1+j\omega/10}{j\omega(1+j\omega/100)}$$



The magnitude response at $\omega = 30$ rad/sec is about – 20 dB = 0.1.

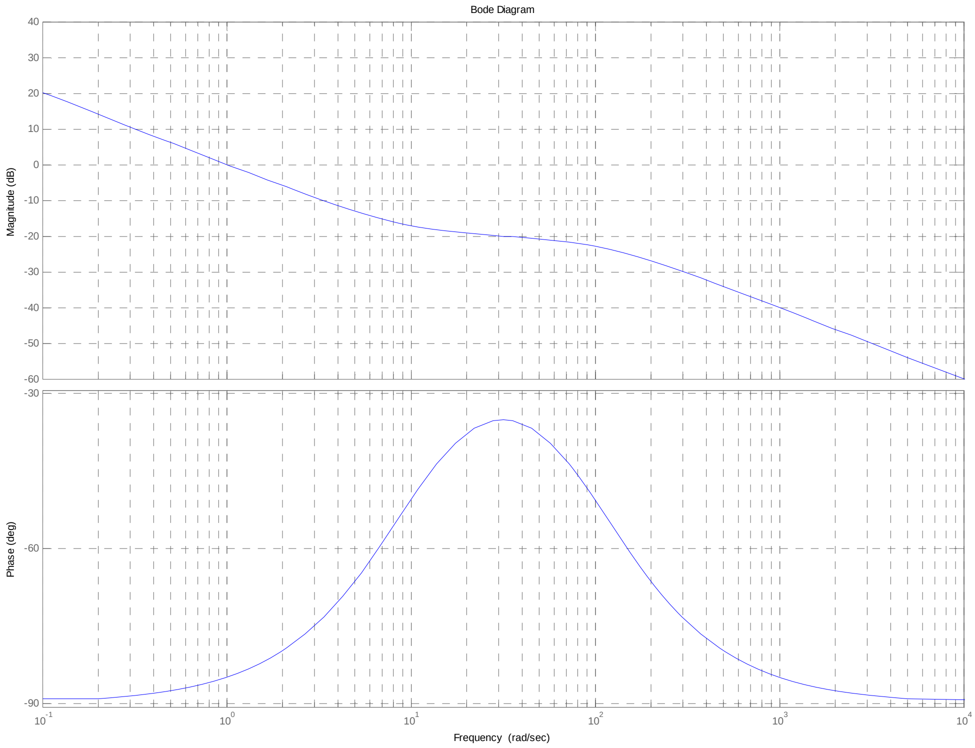


The phase response at $\omega = 30$ rad/sec is about – 45°.

Thus, the output $y(t)$ produced by $u(t) = 5\cos(30t + 30°)$ is roughly given by

$$y(t) = 0.1 \times 5\cos(30t + 30° - 45°) = 0.5\cos(30t - 15°)$$

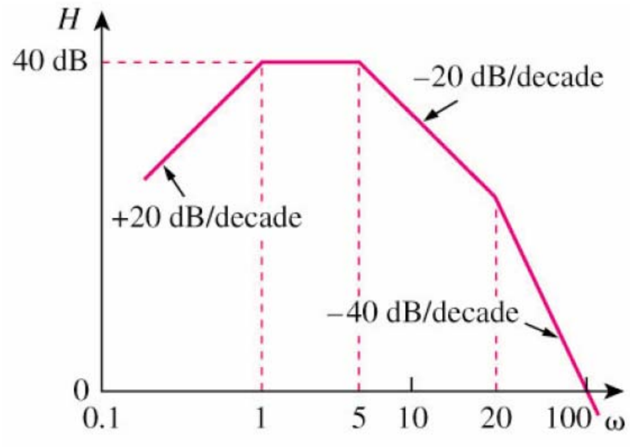
The actual Bode plots of the system generated by MATLAB is given by



Thus, the actual output $y(t)$ produced by $u(t) = 5\cos(30t + 30°)$ is given by

$$y(t) = 0.1 \times 5\cos(30t + 30° - 35°) = 0.5\cos(30t - 5°)$$

Q.2. A Bode plot of $H(j\omega)$ is given in the figure below. Obtain the transfer function $H(s)$.



Solution:

To obtain $H(\omega)$ from the Bode plot, we keep in mind that a zero always cause an upward turn at a corner frequency, while a pole causes a downward turn. We notice from Fig.4 that there is a zero $j\omega$ at the origin which should have intersected the frequency axis at $\omega = 1$. This is indicated by the straight line with slope +20dB/decade. The fact that this straight line is shifted by 40dB indicates that there is a 40-dB gain; that is

$$40 = 20\log_{10} K \Rightarrow \log_{10} K = 2$$

Or

$$K = 10^2 = 100.$$

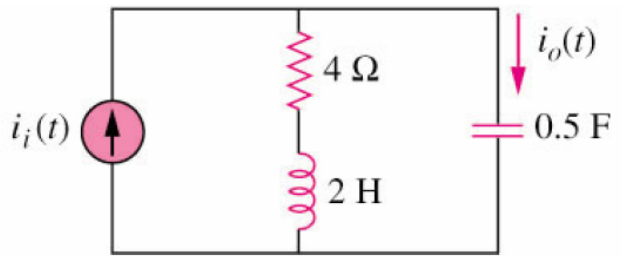
In addition to the zero $j\omega$ at the origin, we notice that there are three factors with corner frequencies at $\omega = 1$,5, and 20 rad/s. Thus, we have:

1. A pole at $p = 1$ with slope –20dB/decade to cause a downward turn and counteract the pole at the origin. The pole at $p = 1$ is determined as $1/(1 + j\omega/1)$.
2. Another pole at $p = 5$ with slope –20dB/decade causing a downward turn. The pole is $1/(1 + j\omega/5)$.
3. A third pole at p = 20 with a slope of –20dB/decade causing a further downturn. The pole is $1/(1 + j\omega/20)$.

Putting all these together gives the corresponding transfer function as

$$H(j\omega) = \frac{100 j\omega}{(1 + j\omega/1)(1 + j\omega/5)(1 + j\omega/20)} \quad \Rightarrow \quad H(s) = \frac{10^4 s}{(s+1)(s+5)(s+20)}$$

Q.3. For the circuit below, obtain the transfer function $I_o(s)/I_i(s)$ and its poles and zeros.



Solution: By current division,

$$I_0(\omega) = \frac{4 + j2\omega}{4 + j2\omega + 1/j0.5\omega} I_i(\omega)$$

or

$$\frac{I_0(\omega)}{I_i(\omega)} = \frac{j0.5\omega(4 + j2\omega)}{1 + j2\omega + (j\omega)^2} = \frac{s(s+2)}{s^2 + 2s + 1}, \quad s = j\omega$$

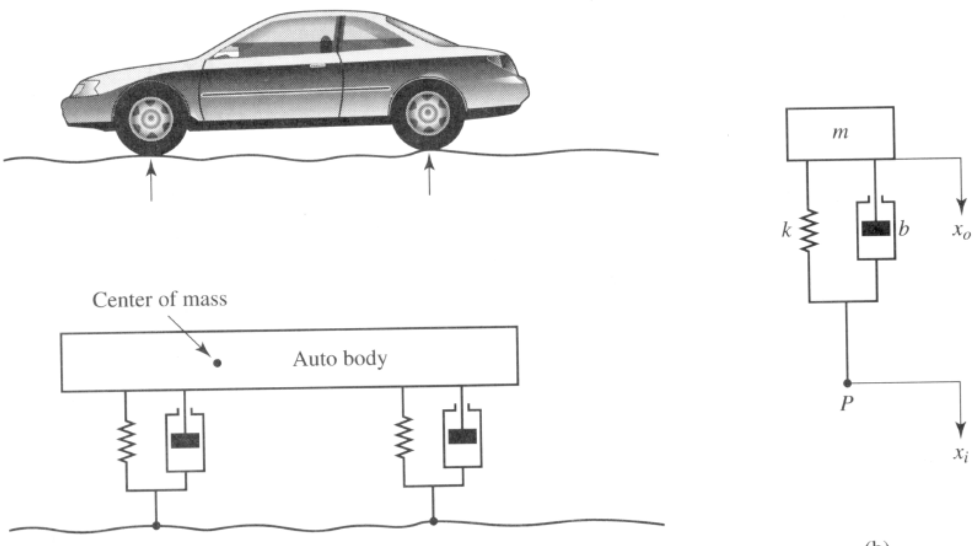
The system zeros are at

$$s(s+2) = 0 \Rightarrow z_1 = 0, z_2 = -2$$

The system poles

$$s^2 + 2s + 1 = (s+1)^2 = 0 \Rightarrow p_1, p_2 = -1$$

Q.4. A car suspension system and a very simplified version of the system are shown in Figures (a) and (b), respectively.

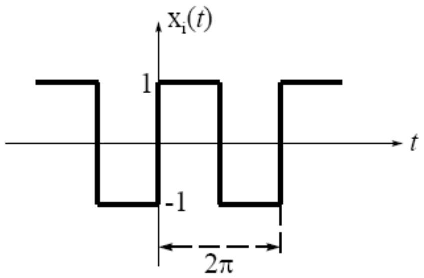


(a) (b)

The transfer function of the simplified car suspension system is

$$G(s) = \frac{bs + k}{ms^2 + bs + k}$$

Suppose a toy car (m = 1 kg, k = 1 N/m and b = 1.414 N s / m) is traveling on a road that has speed reducing stripes and the input to the simplified car suspension system, x_i, may be modeled by the periodic square wave, of frequency ω = 1 rad/s, shown in Figure below.



Determine the steady-state displacement of the car body, $x_{o,ss}(t)$.

Hint : The Fourier Series representation of the periodic square wave shown in Figure above is

$$x_i(t) = \frac{4}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right]$$

Solution:

Question states that the input signal due to the speed reducing strips on the road, $x_i(t)$, may be approximated by the following Fourier Series representation

$$x(t) = \frac{4}{\pi}\left[\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \dots\right]$$

where $\omega = 1$ rad/s.
Since the input consists of 3 sinuosoidal waveforms ($\sin t$, $\sin 3t$ and $\sin 5t$) and system is linear, principle of superposition may be used to determine the solution i.e.

- Find the outputs when the inputs are the sinusoidal waveforms $\sin(\omega_1 t)$ when $\omega_1 = 1, 3, 5$ rad/s
- The output when the input is the periodic square wave is the sum of the output due to the 3 sinusoidal waveforms

Given that $m = 1$ kg, $k = 1\frac{\text{N}}{\text{m}}$ and $b = \sqrt{2}\frac{\text{N}}{\text{m/s}}$, the magnitude and phase of

$$G(j\omega_1) = \frac{j\sqrt{2}\omega_1 + 1}{(j\omega_1)^2 + j\sqrt{2}\omega_1 + 1}$$

when $\omega_1 = 1$ rad/s, 3 rad/s and 5 rad/s are tabulated in the following table

ω_1 (rad/s)	$\lvert G(j\omega_1)\rvert$	$\angle G(j\omega_1)$ (rad)
1	1.2247	-0.6155
3	0.4814	-1.3147
5	0.2854	-1.4248

Hence, the steady-state output is

$$\begin{aligned} x_{o,ss}(t) &= \frac{4}{\pi}\left[1.2247\sin(t - 0.6155) + \frac{0.4814}{3}\sin(3t - 1.3147) + \right. \\ &\qquad \left. \frac{0.2854}{5}\sin(5t - 1.4248) + \dots\right] \\ &= \frac{4}{\pi}[1.2247\sin(t - 0.6155) + 0.1605\sin(3t - 1.3147) + \\ &\qquad 0.005708\sin(5t - 1.4248) + \dots] \end{aligned}$$

Q.5. Consider the second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

whose unit step response has a transient behavior described by the following parameters:

- Rise time, $t_r = 1.8/\omega_n$
- 2% settling time, $t_s = 4/(\zeta\omega_n)$
- Overshoot peak, $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

Sketch and shade the allowable region in the s-plane for the system poles if the step response requirements are

$$t_r < 0.9 \text{ seconds}, \quad t_s < 3 \text{ seconds}, \quad M_p < 10\%$$

Solution:

Desired step response specification are

- $t_r < 0.9$ seconds
- $t_s < 3$ seconds
- $M_p < 10\%$

Rise time, t_r is given by $\frac{1.8}{\omega_n}$. Hence,

$$\frac{1.8}{\omega_n} < 0.9 \qquad \Longrightarrow \omega_n > 2$$

Line of constant ω_n is a semi-circle of radius ω_n, centred at the origin with the two end-points on the imaginary axis. For $\omega_n > 2$, the poles must lie in the LHP and outside a semi-circle of radius 2.

Since 2% settling time $t_s = \frac{4}{\zeta\omega_n}$, the constrain

$$\frac{4}{\zeta\omega_n} < 3 \quad \Rightarrow \quad \zeta\omega_n > \frac{4}{3}$$

As the poles of a prototype 2nd order system are $s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$, the constraint $\zeta\omega_n > \frac{4}{3}$ is satisfied only if the real part of the poles is less than $-\frac{4}{3}$.

Finally, the maximum overshoot should be less than 10% i.e.

$$\begin{aligned}
e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} &< 0.1 \\
\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} &< \ln 0.1 \\
-\pi\zeta &< \ln 0.1\ \sqrt{1-\zeta^2} \\
\pi^2\zeta^2 &> (\ln 0.1)^2(1-\zeta^2) \\
\left[\pi^2 + (\ln 0.1)^2\right]\zeta^2 &> (\ln 0.1)^2 \\
\zeta &> \sqrt{\frac{(\ln 0.1)^2}{\pi^2 + (\ln 0.1)^2}} \\
\zeta &> 0.59
\end{aligned}$$

Poles with the same damping ratio lie on a ray that is rotated $\cos^{-1}\zeta$ from the negative real axis.

Combining the three constraints, the region in the s-plane where the poles may lie in order to satisfy the design specification is found and shown in Figure below.