Q.3 Evaluate the following integrals

(a) $\int_C \frac{1}{z-(1+i)}\,dz, \quad C=\left\{z(t)=e^{it}, \quad 0\le t\le 2\pi\right\}$

(b) $\int_C \frac{1}{z(z-1)}\,dz, \quad C=\left\{z(t)=\tfrac{1}{2}+e^{it}, \quad 0\le t\le 2\pi\right\}$

(c) $\int_0^{2\pi} \frac{1}{3\cos\theta+5}\,d\theta$

(d) $\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx$

(25 marks)

Solution:

(11) Evaluate the following integrals

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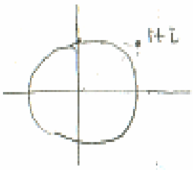
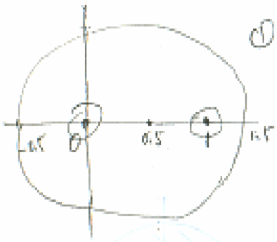
a.) $\int_C \frac{1}{z-(1+i)}\,dz$ $\quad C(t)=(\cos t, \sin t)$ $\quad 0\le t\le 2\pi$



$=0$

b.) $\int_C \frac{1}{z(z-1)}\,dz$ $\quad C(t)=\frac{1}{2}+(\cos t, \sin t)$ $\quad 0\le t\le 2\pi$



① $\int_C \frac{1}{z(z-1)}\,dz = \int_C \frac{dz}{z-1} - \int_C \frac{dz}{z}$

$= 2\pi i - 2\pi i$

$= 0$

c.) $\int_0^{2\pi} \frac{1}{3\cos\theta+5}\,d\theta$

LET $z=(\cos\theta, \sin\theta)=e^{i\theta}$ $\quad \frac{1}{z}=e^{-i\theta}=(\cos\theta, -\sin\theta)$

$2\cos\theta = z+\frac{1}{z} \Rightarrow \cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$

$dz = ie^{i\theta}d\theta$ $\quad d\theta=\frac{1}{i}e^{-i\theta}dz=\frac{1}{i}\cdot\frac{1}{z}dz$

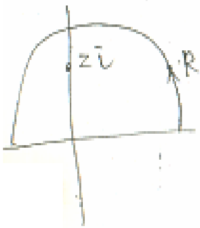
$\therefore \int_0^{2\pi}\frac{1}{3\cos\theta+5}\,d\theta = \int_C \frac{1}{3\times\frac{1}{2}(z+\frac{1}{z})+5}\cdot\frac{1}{iz}\,dz$

$=\frac{1}{i}\int_C \frac{2}{3z^2+10z+3}\,dz = \frac{1}{i}\int_C \frac{2\,dz}{(z+3)(3z+1)} = \frac{2}{3i}\int_C \frac{\frac{1}{z+3}}{z+\frac{1}{3}}\,dz$

$= \frac{2}{3i}\cdot 2\pi i\,\frac{1}{-\frac{1}{3}+3} = \frac{\pi}{2}$

d.) $\int_0^{\infty} \frac{1}{(x^2+4)^2}\,dx$

$\int_0^{\infty}\frac{1}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{(x^2+4)^2}\,dx = \pi i\,\mathrm{Res}(f, 2i) = \pi i\cdot\frac{1}{32i} = \frac{\pi}{32}$



$f(z)=\frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2(z-2i)^2}$

$\mathrm{Res}(f, 2i) = \lim_{z\to 2i}\frac{d}{dz}(z-2i)^2\cdot\frac{1}{(z+2i)^2(z-2i)^2} = \lim_{z\to 2i}\frac{-2}{(z+2i)^3}$

$= \frac{-2}{(4i)^3} = \frac{1}{32i}$

Q.4 A factory manufactures two types of hobby choppers, *A* and *B*, each of which requires labour, machine time and storage space. The two choppers generate net profits of \$500 for *A* and \$650 for *B* per piece, respectively. The factory has a maximum of 300 cm^3 storage space available; chopper *A* has a volume of 3 cm^3 and chopper *B* has a volume of 5 cm^3. The factory has a maximum of 500 machine hours per day to manufacture the helicopters and the choppers *A* and *B* require 8 and 10 machine hours, respectively. The factory has a maximum of 160 labour hours per day to manufacture the choppers and the choppers *A* and *B* require 2 and 3 labour hours, respectively. Let *x* and *y* represent the number of choppers *A* and *B* produced daily. Use the Simplex Method to maximize the daily profit.

(25 marks)

Solution: It is straightforward to formulate the problem as maximizing

$$P = 500x + 650y$$

subject to the following constraints

$$\begin{aligned} 3x + 5y &\le 300 \\ 8x + 10y &\le 500 \\ 2x + 3y &\le 160 \\ x \ge 0, \quad y &\ge 0 \end{aligned}$$

Introducing slack variables w_1, w_2 and w_3, the constraints then become

$$\begin{aligned} 3x + 5y + w_1 &= 300 \\ 8x + 10y + w_2 &= 500 \\ 2x + 3y + w_3 &= 160 \\ x \ge 0, \quad y \ge 0, \quad w_1 \ge 0, \quad w_2 &\ge 0 \end{aligned}$$

And the objective function

$$P - 500x - 650y = 0$$

From this, we set up the simplex table for the problem.

Basis	x	y	w_1	w_2	w_3	b	Check
W_1	3	5	1	0	0	300	309
w_2	8	10	0	1	0	500	519
w_3	2	3	0	0	1	160	166
P	− 500	− 650	0	0	0	0	− 1150

Basis	x	y	w_1	w_2	w_3	b	Check
W_1	3	5	1	0	0	300	309
y ~~w_2~~	0.8	1	0	0.1	0	50	51.9
w_3	2	3	0	0	1	160	166
P	− 500	− 650	0	0	0	0	− 1150

Basis	x	y	w_1	w_2	w_3	b	Check
w_1	− 1	0	1	− 0.5	0	50	49.5
y ~~w_2~~	0.8	1	0	0.1	0	50	51.9
w_3	− 0.4	0	0	− 0.3	1	10	10.3
P	20	0	0	65	0	32500	32585

Optimal solution: $x = 0, \quad y = 50, \quad P_{\max} = \$32,500$