

**Q.3 (a)** Given a complex function

$$f(z) = \frac{2z+3}{e^z-1}$$

determine its singular points, the orders of singularise of its singular points and the corresponding residues.

**Solution:** Noting that

$$e^z = 1 + z + \frac{z^2}{2!} + \dots \Rightarrow e^z - 1 = z + \frac{z^2}{2!} + \dots = z \left( 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots \right)$$

it follows that  $f(z)$  has a **1st order singular point** at  $z=0$ , and the corresponding **residue** is given by

$$\text{Res}(f, 0) = \lim_{z \rightarrow 0} \frac{2z+3}{(e^z-1)'} = \lim_{z \rightarrow 0} \frac{2z+3}{e^z} = 3$$

(6 marks)

(b) Given a complex function

$$f(z) = ze^{1/z}$$

determine its singular points, the orders of singularise of its singular points and the corresponding residues.

**Solution:** Noting that

$$f(z) = ze^{1/z} = z \left( 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots \right) = z + 1 + \frac{1}{2!z} + \frac{1}{3!z^2} + \dots$$

it follows  $f(z)$  has **an essential singularity** at  $z=0$ , and by definition, its **residue** is given by

$$\text{Res}(f, 0) = a_{-1} = \frac{1}{2}$$

(6 marks)

(c) Evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^2} dx$$

**Solution:** Let

$$f(z) = \frac{1}{(z^2 + 1)^2} \Rightarrow \text{its has a 2nd order singularity at } z = i \text{ on the upper plane}$$

We have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{(x^2 + 1)^2} dx &= 2\pi i \operatorname{Res}(f, i) = 2\pi i \frac{1}{(2-1)!} \lim_{z \rightarrow i} \frac{d}{dz} \left[ (z-i)^2 \frac{1}{(z^2 + 1)^2} \right] \\ &= 2\pi i \lim_{z \rightarrow i} \frac{d}{dz} \left[ \frac{1}{(z+i)^2} \right] = 2\pi i \lim_{z \rightarrow i} \frac{-2}{(z+i)^3} = 2\pi i \cdot \frac{1}{4i} \\ &= \frac{\pi}{2} \end{aligned}$$

(6 marks)

(d) Given a complex function  $f(z) = \frac{1}{(z-1)(z-2)}$ ,  $C_1 : |z|=1$  and  $C_2 : |z|=2$ , find the

Laurent series for  $f(z)$  in the region  $1 < |z| < 2$  as indicated in Fig. Q.3 (d).

**Solution:**

(7 marks)

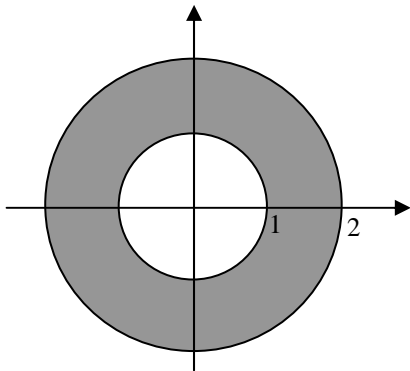


Fig. Q.3 (d)

$$\begin{aligned} f(z) &= \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \\ &= -\frac{1}{2} \cdot \frac{1}{1 - z/2} - \frac{1}{z} \cdot \frac{1}{1 - 1/z} \\ &= -\frac{1}{2} \left( 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots \right) - \frac{1}{z} \left( 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right) \\ &= -\left( \frac{1}{z} + \frac{1}{z^2} + \dots \right) - \frac{1}{2} - \left( \frac{z}{2^2} + \frac{z^2}{2^3} + \dots \right) \end{aligned}$$

Alternatively, one may solve this problem with

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz = \frac{1}{2\pi i} \oint_C \frac{1}{(z-1)(z-2)z^{n+1}} dz$$

where  $C$  is any close path within the ring. The solution should be the same as given above.

**Q.4** Refer to Fig. Q.4 below. The upper arc between  $z_2$  and  $z_1$  is kept at a potential of  $-3000\text{ V}$  whereas the lower arc is kept at a potential of  $3000\text{ V}$ .

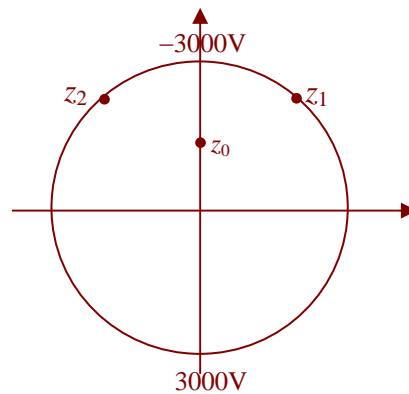


Fig. Q.4

- (a) Determine a linear fractional transformation  $w = g(z)$  that maps  $|z| \leq 1$  onto  $|w| \leq 1$  (i.e., a unit disc onto another unit disc) with  $z_0 = i/2$  being mapped onto  $w = 0$ . Verify that  $z_1 = (3+i4)/5$  is mapped onto  $w = -1$ , and  $z_2 = (-3+i4)/5$  is mapped onto  $w = 1$ .

**Solution:** Recall the mapping that maps a unit disc onto another disc in the lecture notes, we have

$$w = \frac{z - z_0}{z_0^* z - 1} = \frac{z - i/2}{-iz/2 - 1} = \frac{2z - i}{-iz - 2} = \frac{i - 2z}{2 + iz}$$

For  $z_1 = (3+i4)/5$ ,

$$w_1 = \frac{i - 2z_1}{2 + iz_1} = \frac{i - 2(3+i4)/5}{2 + i(3+i4)/5} = \frac{5i - 6 - 8i}{10 + 3i - 4} = \frac{-6 - 3i}{6 + 3i} = -1$$

For  $z_2 = (-3+i4)/5$ ,

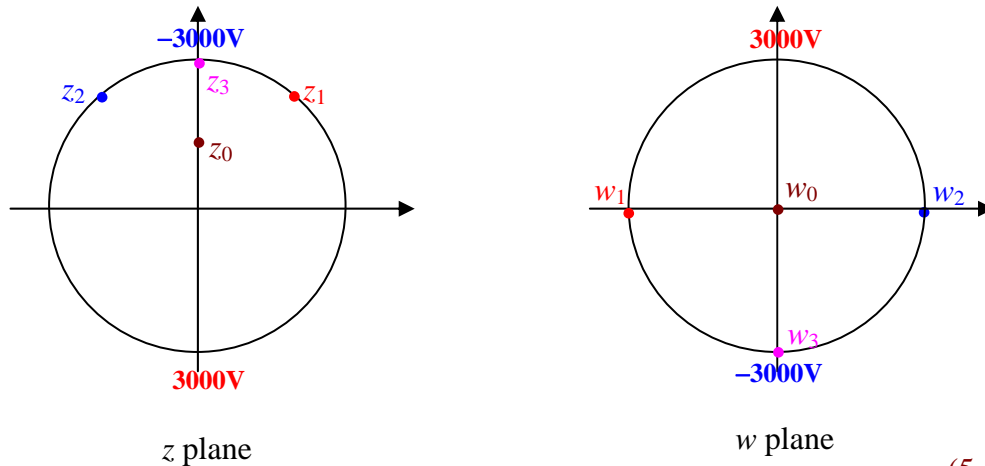
$$w_2 = \frac{i - 2z_2}{2 + iz_2} = \frac{i - 2(-3+i4)/5}{2 + i(-3+i4)/5} = \frac{5i + 6 - 8i}{10 - 3i - 4} = \frac{6 - 3i}{6 - 3i} = 1$$

(10 marks)

- (b) Plot the image of the unit circle on the  $w$  plane and label the potentials of the arcs after transformation.

**Solution:** We need to determine how the upper arc between  $z_2$  and  $z_1$  are mapped. Let us

choose another point  $z_3 = i$ . We have  $w_3 = \frac{i - 2z_3}{2 + iz_3} = \frac{i - 2i}{2 + i^2} = \frac{-i}{1} = -i$ . Thus,



(5 marks)

- (c) Determine the electrostatic potential function inside the unit disc of Fig. Q.4. What is the potential at  $z_0 = i/2$ ?

**Solution:** The problem under transformation is exactly the same as that of Example 8 in the lecture notes. Recall the mapping obtained in Example 8, i.e.,  $w = -i \cdot \frac{z+1}{z-1}$ . Thus, the required transformation that transform the given problem into **Example 3** of **Simple Geometries** is given by

$$w' = -i \cdot \frac{w+1}{w-1} = -i \frac{\frac{i-2z}{2+iz} + 1}{2 + \frac{i-2z}{2+iz}} = -i \frac{i-2z+2+iz}{4+i2z+i-2z} = \frac{(z+1)+i2(z-1)}{(4-2z)+i(1+2z)}$$

and thus the complex and real potential functions are respectively given by

$$F(w) = -i \frac{6000}{\pi} \ln|w'| + \frac{6000}{\pi} \arg(w') - 3000$$

$$\Phi(x, y) = \text{Re} \left[ F \left( w' = \frac{(z+1)+i2(z-1)}{(4-2z)+i(1+2z)} \right) \right] = \frac{6000}{\pi} \arg \left( \frac{(z+1)+i2(z-1)}{(4-2z)+i(1+2z)} \right) - 3000$$

The potential at  $z_0 = i/2$ , which is mapped onto  $w_0 = 0$ , is **0**.

(10 marks)