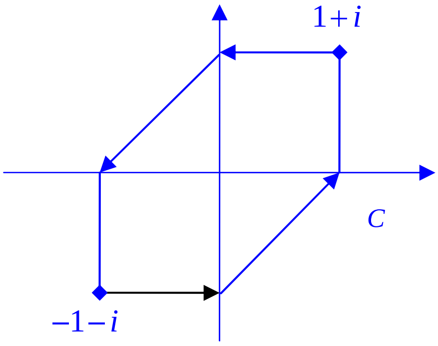
Q.3 (a) Evaluate $\int_C e^{z+i-1}\,dz$, where C is given as in the figure below:



(6 marks)

Solution: $\int_C e^{z+i-1}\,dz = 0$ because e^{z+i-1} is analytic everywhere on the complex plane.

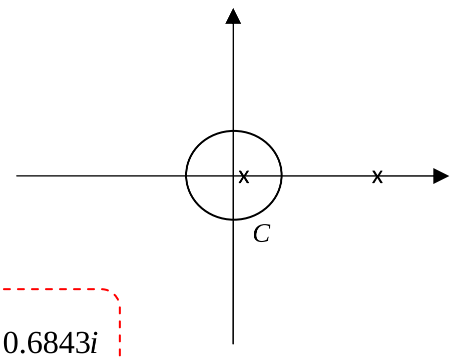
(b) Evaluate $\displaystyle\oint_{|z|=1/30} \frac{z+1/30}{\left(z+1/3\right)\left(z-1/300\right)}\,dz$

(6 marks)

Solution: There is one singular point $z_0 = \dfrac{1}{300}$ encircled by the integration curve C. Thus,

$$\operatorname{Res}(f, z_0) = \operatorname{Res}\left(f, \frac{1}{300}\right)$$

$$= \lim_{z\to 1/300} \frac{z+1/30}{z+1/3} = \frac{1/300+1/30}{1/300+1/3} = \frac{11}{101}$$

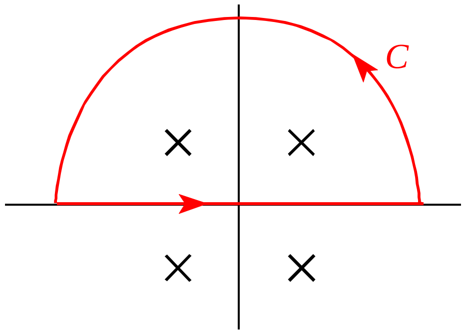


$$\oint_{|z|=1/30} \frac{z+1/30}{\left(z+1/3\right)\left(z-1/300\right)}\,dz = 2\pi i\frac{11}{101} = \frac{22\pi i}{101} = 0.2178\pi i = 0.6843i$$

(c) Use the complex integration approach to compute $\int_0^\infty \frac{x^2}{x^4+1}dx$

(6 marks)

Solution: Let $f(z)=\frac{z^2}{z^4+1}$. Its poles can be computed as



$$z^4+1=0 \quad\Rightarrow\quad z^4=-1=e^{(2k\pi-\pi)i}, \quad i=0,1,2,3,\cdots$$

$$z_1=e^{-\frac{\pi}{4}i}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i, \qquad z_2=e^{\frac{\pi}{4}i}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$$

$$z_3=e^{\frac{3\pi}{4}i}=-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i, \quad z_4=e^{\frac{5\pi}{4}i}=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i$$

$$\int_0^\infty \frac{x^2}{x^4+1}dx=\frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+1}dx=\frac{1}{2}\cdot 2\pi i\left[\mathrm{Res}\left(f,\frac{\sqrt{2}}{2}(1+i)\right)+\mathrm{Res}\left(f,\frac{\sqrt{2}}{2}(-1+i)\right)\right]$$

$$=\pi i\left\{\left[\frac{1}{4z}\right]_{z=\frac{\sqrt{2}}{2}(1+i)}+\left[\frac{1}{4z}\right]_{z=\frac{\sqrt{2}}{2}(-1+i)}\right\}$$

$$=\pi i\left[\frac{1}{2\sqrt{2}(1+i)}+\frac{1}{2\sqrt{2}(-1+i)}\right]=\frac{\pi i}{2\sqrt{2}}\left[\frac{1}{1+i}+\frac{1}{-1+i}\right]=\boxed{\frac{\pi}{2\sqrt{2}}=0.3536\pi=1.1107}$$

(d) Use the complex integration approach to compute $\int_0^\infty \frac{\cos x}{(x^2+1)^2}dx$

(7 marks)

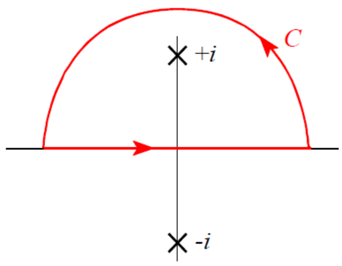
Solution: Let $f(z)=\frac{1}{\left(z^2+1\right)^2}=\frac{1}{(z+i)^2(z-i)^2}$.

$$\int_0^\infty \frac{\cos x}{(x^2+1)^2}dx=\frac{1}{2}\int_{-\infty}^\infty \frac{\cos x}{(x^2+1)^2}dx=\frac{1}{2}\mathrm{Re}\left[2\pi i\cdot\mathrm{Res}\left(fe^{iz},i\right)\right]$$

$$=\mathrm{Re}\left[\pi i\cdot\lim_{z\to i}\frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right)\right]$$

$$=\mathrm{Re}\left[\pi i\cdot\left(\frac{ie^{iz}}{(z+i)^2}-\frac{2e^{iz}}{(z+i)^3}\right)_{z=i}\right]$$

$$=\boxed{\frac{\pi}{2e}=0.5779}$$



Q.4 Consider a real-valued function

$$f(\mathbf{x}) = f(x_1, x_2) = 1 + (\cos x_1)^2 + 2x_2 \cos x_1 + (x_2)^2$$

(a) Use the Cauchy's Method of Steepest Descent to design an iteration scheme to determine the minimum of $f(\mathbf{x})$.

(9 marks)

Solution: Given the function, its gradient can be computed as follows

$$\nabla f(\mathbf{x}) = \begin{pmatrix} -2\cos x_1 \sin x_1 - 2x_2 \sin x_1 \\ 2\cos x_1 + 2x_2 \end{pmatrix} = 2(\cos x_1 + x_2)\begin{pmatrix} -\sin x_1 \\ 1 \end{pmatrix}$$

Thus, we can design an iteration scheme based on the Cauchy's Method of Steepest Descent as follows:

$$\mathbf{x}_i = \mathbf{x}_{i-1} - t^* \nabla f(\mathbf{x}_{i-1}) = \begin{pmatrix} x_{1,i-1} \\ x_{2,i-1} \end{pmatrix} - 2t^*(\cos x_{1,i-1} + x_{2,i-1})\begin{pmatrix} -\sin x_{1,i-1} \\ 1 \end{pmatrix}$$

(b) Let the step size be fixed with $t^* = 0.5$ and the initial point $\mathbf{x}_0 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$. Calculate the iteration scheme obtained in Part (a) up to 5 steps to obtain an approximation of the minimum of $f(\mathbf{x})$. Give the minimum value obtained.

(8 marks)

Solution: We summarize the iterative calculation in the following table:

Step i	$\mathbf{x}_i$	
	$x_{1,i}$	$x_{2,i}$
0	–1	–2
1	0.2283	–0.5403
2	0.3265	–0.9741
3	0.3178	–0.9472
4	0.3187	–0.9499
5	0.3186	–0.9496

$$f(\mathbf{x}) = 1 + (\cos x_1)^2 + 2x_2 \cos x_1 + (x_2)^2 = 1 + (\cos 0.3186)^2 - 2 \times 0.9496 \cos 0.3186 + (-0.9496)^2 = 1$$

(c) Repeat Part (b) with the same fixed step, but with a new initial point $\mathbf{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Comment on the result you have obtained.

(8 marks)

Solution: We summarize the iterative calculation in the following table:

Step i	$\mathbf{x}_i$	
	$x_{1,i}$	$x_{2,i}$
0	1	2
1	3.1314	1
2	3.1314	0.9999
3	3.1314	0.9999
4	3.1314	0.9999
5	3.1314	0.9999

$$f(\mathbf{x}) = 1 + (\cos x_1)^2 + 2x_2 \cos x_1 + (x_2)^2 = 1 + (\cos 3.1314)^2 + 2 \times 0.9999 \cos 3.1314 + (0.9999)^2 = 1$$

Comments:

1) the problem has multiple solutions, and
2) the convergent rate corresponding to the 2nd initial point is much faster.