Q.3 (a) Determine all the possible values for $\ln(-1)$ and $\ln(2012)$.

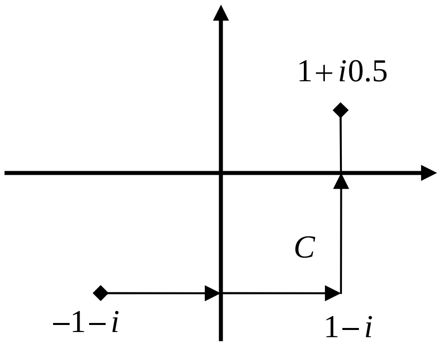
(5 marks)

Solution: By definition

$$\ln(-1) = \ln|-1| + i\arg(-1) = 0 + i(2k+1)\pi = i(2k+1)\pi, \quad k = 0, \pm 1, \pm 2, \cdots$$

$$\ln(2012) = \ln|2012| + i\arg(2012) = 7.6069 + i2k\pi, \quad k = 0, \pm 1, \pm 2, \cdots$$

(b) Compute $\int_C z\,dz$, where C is a curve connecting point $(-1-i)$ to point $(1-i)$ and then finally to point $(1+i0.5)$, as showed in the figure below.



(5 marks)

Solution: Let C_1 be the straight line connecting points from $-1-i$ to $1-i$. We have

$$z(t) = (2t-1) + i(-1) = x(t) + iy(t), \quad t \in [0,1] \quad \Rightarrow \quad z'(t) = x'(t) + iy'(t) = 2$$

Let C_2 be the straight line connecting points from $1-i$ to $1+i0.5$. We have

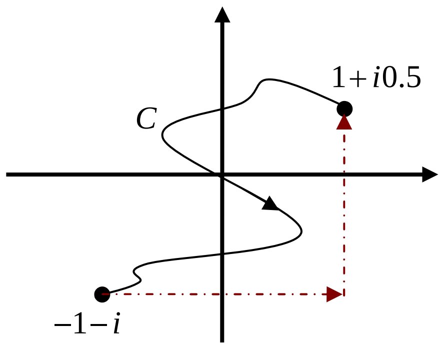
$$z(t) = 1 + i(1.5t - 1) = x(t) + iy(t), \quad t \in [0,1] \quad \Rightarrow \quad z'(t) = x'(t) + iy'(t) = i1.5$$

Thus,

$$\begin{aligned}
\int_C z\,dz &= \int_{C_1} z\,dz + \int_{C_2} z\,dz \\
&= \int_{C_1} z(t)z'(t)\,dt + \int_{C_2} z(t)z'(t)\,dt \\
&= \int_0^1 (2t-1-i)2dt + \int_0^1 (i1.5t+1-i)i1.5dt \\
&= \left(2t^2 - 2t - i2t\right)\Big|_0^1 + \left(-1.125t^2 + 1.5t + i1.5t\right)\Big|_0^1 \\
&= 2 - 2 - i2 - 1.125 + 1.5 + i1.5
\end{aligned}$$

$$\int_C z\,dz = 0.375 - i0.5$$

(c) Compute $\int_C z\,dz$, where C is a curve connecting point $(1+i0.5)$ to point $(-1-i)$, as showed in the figure below.



(5 marks)

Solution: Since $f(z)=z$ is analytic everywhere on the complex plane, by Cauchy's integral theorem, the integration of the function on the closed path as depicted in the figure above is 0, i.e.,

$$\int_{C+\text{ dashed lines}} z\,dz = \int_C z\,dz + \int_{\text{dashed lines}} z\,dz = 0$$

$$\Downarrow$$

$$\int_C z\,dz = -\int_{\text{dashed lines}} z\,dz = -0.375+i0.5$$

where $\int_{\text{dashed lines}} z\,dz$ is exactly the complex integration in Part (c).

(d) Compute $\oint_{|z|=\pi} \frac{1}{z-3+i}\,dz$ (5 marks)

Solution: $$\oint_{|z|=\pi} \frac{1}{z-3+i}\,dz = 0$$

(e) Compute $\oint_{|z|=\pi} \frac{1}{z-2+i2}\,dz$ (5 marks)

Solution: $$\oint_{|z|=\pi} \frac{1}{z-2+i2}\,dz = 2\pi i \cdot \text{Res}(f, 2-i2) = 2\pi i \lim_{z\to 2-i2}(z-2+i2)\frac{1}{z-2+i2} = 2\pi i$$

Q.4 (a) Why are Cauchy-Rieman equations required for a complex function of complex variable to be analytic?

(4 marks)

<u>Solution:</u> Given a complex function, $f(z) = u(x, y) + iv(x, y)$, the Cauchy-Rieman equations, i.e.,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are required for the function *to force its derivatives in every direction on the complex plane to be identical*. It thus has a unique derivative, which is said to be analytic.

(b) Comment why the coefficient a_{-1} in the Laurent series of a complex function $f(z)$, i.e.,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is so special in complex integration as compared to all the other coefficients.

(5 marks)

<u>Solution:</u> By noting the Laurent series of a complex function $f(z)$, i.e.,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

which implies

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz \quad \Rightarrow \quad \oint_C f(z) dz = 2\pi i \cdot a_{-1}$$

It is special because it is directly related to the complex integral of the given function.

(c) What is the Cauchy's method of steepest descent for? Comment on the key idea behind the Cauchy's method of steepest descent.

(8 marks)

Solution:

- The Cauchy's method of steepest descent was developed to solve *unconstrained optimization problems*, i.e., to find either the minima (or maxima) of a nonlinear function.

(4 marks)

- The key idea behind the Cauchy's method of steepest descent is to obtain *an iteration scheme searching the minima along the direction of negative gradient of the function*. By appropriately choosing a step size, it is guaranteed that the iteration is getting closer and closer to the target.

(4 marks)

(d) What is the linear programming technique or the simplex method for? Comment on the key idea behind the simplex method.

(8 marks)

Solution:

- The linear programming technique or the simplex method was developed to solve *constrained optimization problems*, i.e., to maximize (or minimize) a linear index function with a certain set of linear constraints.

(4 marks)

- The key idea behind the simplex method is *to convert the constrained problem into the solution of a set of linear equations (simplex table)* and then use the *basic or elementary operations* involved in solving linear equations (such as multiplication of an equation by a nonzero constant, and addition of two equations, etc.) to yield a desired solution.

(4 marks)