Q.3 (a) Express $(1+i)^{400}$ in the form of $x+iy$.

(5 marks)

Solution:

$$
\begin{aligned}
(1+i)^{400} &= \left(\sqrt{2}e^{j\frac{\pi}{4}}\right)^{400} \\
&= \left(\sqrt{2}\right)^{400} e^{j\frac{\pi}{4}\times 400} \\
&= 2^{200} e^{j100\pi} \\
&= 2^{200}(\cos 100\pi + j\sin 100\pi) \\
&= 2^{200}(1+j0) \\
&= 2^{200}
\end{aligned}
$$

(b) Give the Taylor series expansion of $\frac{1}{z}$ about $z_0 = i$. Show the first four terms and give the radius of convergence.

(5 marks)

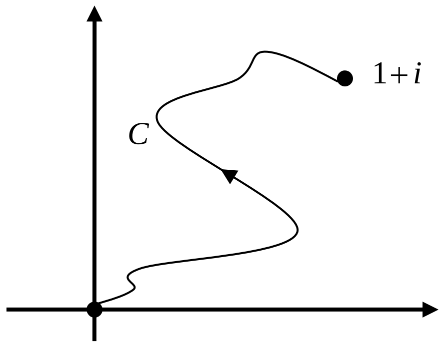
Solution:

$$
\begin{aligned}
\frac{1}{z} &= \frac{-1}{-i-(z-i)} \\
&= \frac{1}{i}\cdot\frac{1}{1-\frac{z-i}{-i}} \\
&= \frac{1}{i}\left[1+\frac{z-i}{-i}+\left(\frac{z-i}{-i}\right)^2+\left(\frac{z-i}{-i}\right)^3+\cdots\right] \\
&= \frac{1}{i}\left[1-\frac{z-i}{i}+\left(\frac{z-i}{i}\right)^2-\left(\frac{z-i}{i}\right)^3+\cdots\right] \\
&= \frac{1}{i}+(z-i)-\frac{1}{i}(z-i)^2-(z-i)^3+\cdots
\end{aligned}
$$

The series converts for all $\left|\frac{z-i}{-i}\right| = \frac{|z-i|}{|-i|} = |z-i| < 1$. Thus,

The radius of convergence = 1.

(c) Compute $\int_C z^2\, dz$, where C is a curve connecting point the origin to point $(1+i)$, as showed in the figure below.



(5 marks)

Solution: Since z^2 is analytic everywhere on the whole complex plane, by Cauchy's integral theorem, $\int_C z^2 dz$ with C as indicated in the figure above is the same as the integration of z^2 along a straight line from the origin to 1 + *i*, which can be characterized by $z(t) = t(1+i), \quad t \in [0,1]$. Thus,

$$\begin{aligned}
\int_C z^2\, dz &= \int_0^1 t^2 (1+i)^2 (1+i) dt \\
&= (1+i)^3 \int_0^1 t^2 dt \\
&= (1+i)^3 \frac{1}{3} t^3 \Big|_0^1 \\
&= \frac{1}{3}(1+i)^3 \\
&= -\frac{2}{3} + i\frac{2}{3}
\end{aligned}$$

(d) Compute $\oint\limits_{|z|=\pi} \sin\left(\frac{1}{z}\right) dz$ (5 marks)

Solution:

$$\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \cdots$$
$$\Downarrow$$
$$f(z) = \sin\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right) - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \frac{1}{5!}\left(\frac{1}{z}\right)^5 - \frac{1}{7!}\left(\frac{1}{z}\right)^7 + \cdots$$
$$\Downarrow$$
$$\text{Res}(f, 0) = 1$$
$$\Downarrow$$
$$\oint\limits_{|z|=\pi} \sin\left(\frac{1}{z}\right) dz = 2\pi i \cdot \text{Res}(f, 0) = 2\pi i$$

(e) Compute $\int_{-\infty}^{\infty} \sin x \cdot \frac{x + \sin 1}{x^2 + 1} dx$ (5 marks)

Solution:

$$\begin{aligned}
\int_{-\infty}^{\infty} \sin x \cdot \frac{x + \sin 1}{x^2 + 1} dx &= \text{Im}\left[2\pi i \sum_j \text{Res}(f\, e^{iz}, a_j)\right] \\
&= \text{Im}\left[2\pi i \cdot \text{Res}\left(\frac{z + \sin 1}{z^2 + 1} e^{iz}, i\right)\right] \\
&= \text{Im}\left[2\pi i \cdot \text{Res}\left(\frac{z + \sin 1}{(z+i)(z-i)} e^{iz}, i\right)\right] \\
&= \text{Im}\left[2\pi i \cdot \frac{z + \sin 1}{(z+i)} e^{iz}\Big|_{z=i}\right] \\
&= \text{Im}\left[2\pi i \cdot \frac{i + \sin 1}{2i} e^{-1}\right] \\
&= \text{Im}\left[\frac{\pi}{e}(i + \sin 1)\right] \\
&= \frac{\pi}{e}
\end{aligned}$$

Q.4 (a) What is the difference between the Cauchy's method of steepest descent and the simplex method?

(5 marks)

Solution:

- Cauchy's method of steepest descent is an **unconstrained** optimization technique that can be utilized to find an optimal solution for a **nonlinear** function.
- The simplex method is a **constrained** optimization technique for finding optimal solution for a **linear** objective function.

(b) Consider an objective function $f(x_1, x_2) = 29x_1^2 + 45x_2^2$. Suggest an optimization technique that can be used to determine its minimum. Give the iterative procedure (algorithm) for computing the minimum.

(**Note that you do not need to compute the minimum)**

(10 marks)

Solution: This is an unconstrained optimization problem and the objective function is nonlinear. Thus, we have to employ the Cauchy's method of steepest descent to solve this problem. The iterative scheme is given by

$$\mathbf{x}_i = \mathbf{x}_{i-1} - t_i \nabla f(\mathbf{x}_{i-1})$$

where t_i is the step size and

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{pmatrix} = \begin{pmatrix} 58x_1 \\ 90x_2 \end{pmatrix}$$

(c) Consider a function $f(x_1, x_2) = 29x_1^2 + 45x_2^2$ with constraints $2x_1^2 + 8x_2^2 \le 60$ and $4x_1^2 + 4x_2^2 \le 60$. Can we use linear programming techniques, such as the simplex method, to find its maximum? Why or why not? State clearly your reasoning.

(**Note that you do not need to compute the maximum)**

(10 marks)

Solution: Although the given objective function is nonlinear in terms of x_1 and x_2, it is linear, however, in terms of $y_1 = x_1^2$ and $y_2 = x_2^2$. The problem can then be translated into an equivalent problem of maximizing the objective function

$$f(y_1, y_2) = 29y_1 + 45y_2$$

subject to the following constraints:

$$2y_1 + 8y_2 \le 60$$

$$4y_1 + 4y_2 \le 60$$

$$y_1 \ge 0$$

$$y_2 \ge 0$$

Thus, the linear programming technique can be used to solve such a problem.