Q.3 (a) Find the derivative of $f(z) = \mathrm{Im}(z)$ if existent. Justify your answer otherwise.

(5 marks)

Solution:

$$f(z) = \mathrm{Im}(z) = y + i \cdot 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = 1 \neq -\frac{\partial v}{\partial y} = 0$$

Thus, $f(z)$ is not differentiable and its derivative does not exist.

(b) Compute $\oint_{|z|=1} \bar{z} \cdot dz$, where $\bar{z}$ is the complex conjugate of z.

(5 marks)

Solution: On the unit circle, $z = e^{i\theta}, 0 \le \theta \le 2\pi$

$$\oint_{|z|=1} \bar{z} \cdot dz = \int_0^{2\pi} e^{-i\theta} \cdot ie^{i\theta} d\theta = i \int_0^{2\pi} d\theta = 2\pi i$$

(c) Compute $\oint_{|z|=\pi} z^2 e^{1/z} dz$

(5 marks)

Solution:

$$f(z) = z^2 e^{1/z} = z^2 \left(1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} + \cdots \right) = z^2 + z + \frac{1}{2!} + \frac{1}{3!} \cdot \frac{1}{z} + \frac{1}{4!} \cdot \frac{1}{z^2} + \cdots$$

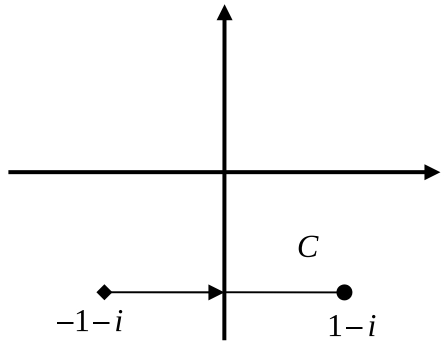
has an essential singularity point at $z_0 = 0$ and its corresponding residue or

$$a_{-1} = \frac{1}{3!} = \frac{1}{6}$$

Thus,

$$\oint_{|z|=\pi} z^2 e^{1/z} dz = 2\pi i \cdot \mathrm{Res}(f, z_0) = \frac{2\pi i}{6} = \frac{\pi}{3} i$$

(d) Compute $\int_C \mathrm{Re}(z)\,dz$, where C is a straight line connecting point $(-1-i)$ to point $(1-i)$, as showed in the figure below.



(5 marks)

Solution: $\mathrm{Re}(z)$ is not an analytic function. We have to integrate it directly. On the straight line *C* connecting point $(-1-i)$ to point $(1-i)$, we have

$$z(t) = (b-a)t + a = (1-i+1+i)t - 1 - i = (2t-1) - i, \quad t \in [0,1]$$

and the integral on *C* is given by

$$\int_C \mathrm{Re}(z)\,dz = \int_0^1 \mathrm{Re}(z(t))z'(t)dt = \int_0^1 (2t-1)2dt = 2(t^2-t)\Big|_0^1 = 0$$

(e) Compute $\int_0^\infty \frac{\cos(\pi x)}{4x^2+1}dx$

(5 marks)

Solution: Since $\frac{\cos(\pi x)}{4x^2+1}$ is an even function, we have

$$\int_0^\infty \frac{\cos(\pi x)}{4x^2+1}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos(\pi x)}{4x^2+1}dx = \frac{1}{2}\mathrm{Re}\left[2\pi i \sum_j \mathrm{Res}\left(fe^{i\pi z}, a_j\right)\right] = \frac{1}{2}\mathrm{Re}\left[2\pi i \cdot \mathrm{Res}\left(\frac{1}{(2z+i)(2z-i)}e^{i\pi z}, \frac{i}{2}\right)\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[2\pi i \cdot \lim_{z\to\frac{i}{2}}\left(z-\frac{i}{2}\right)\frac{1}{(2z+i)(2z-i)}e^{i\pi z}\right] = \frac{1}{2}\mathrm{Re}\left[\pi i \cdot \lim_{z\to\frac{i}{2}}(2z-i)\frac{1}{(2z+i)(2z-i)}e^{i\pi z}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\pi i \cdot \frac{1}{2\frac{i}{2}+i}e^{i\pi\frac{i}{2}}\right] = \frac{1}{2}\mathrm{Re}\left[\pi \cdot \frac{1}{2}e^{-\frac{\pi}{2}}\right] = \frac{\pi}{4}e^{-\frac{\pi}{2}} = 0.1633$$

Q.4 The C-gate Technology has hired three workers, W_1, W_2 and W_3, to produce two kinds of hard disk drives, D_1 and D_2, the profit being \$3 and \$2, respectively.

- Worker W_1 prepares all parts for D_1 in 4 minutes and for D_2 in 3 minutes. W_1 works for 8 hours daily.
- Worker W_2 assembles D_1 in 6 minutes and D_2 in 5 minutes. W_2 works for 8 hours daily.
- Worker W_3 packs both D_1 and D_2 in 1 minute each. W_3 only works for 4 hours daily in the afternoon.

Use the Simplex Method or any other technique to determine production figures that maximize the daily (8 working hours) profit.

(25 marks)

Solution:

If C-gate Technology produces x_1 sets of drive D_1 and x_2 sets of drive D_2 daily (8 hours), the profit daily is

$$f(x_1, x_2) = 3x_1 + 2x_2$$

The constraints are

$$4x_1 + 3x_2 \le 480 \quad \text{(resulting from Worker } W_1 \text{ in 8 hours)}$$

$$6x_1 + 5x_2 \le 480 \quad \text{(resulting from Worker } W_2 \text{ in 8 hours)}$$

$$x_1 + x_2 \le 240 \quad \text{(resulting from Worker } W_3 \text{ in 4 hours)}$$

$$x_1 \ge 0, \quad x_2 \ge 0$$

We introduce slack variables w_1 and w_2. The constraints become

$$4x_1 + 3x_2 + w_1 = 480$$

$$6x_1 + 5x_2 + w_2 = 480$$

$$x_1 + x_2 + w_3 = 240$$

$$x_1, x_2, w_1, w_2\ w_3 > 0$$

and the objective function $f - 3x_1 - 2x_2 = 0$. We construct the following Simplex Table:

basis	x_1	x_2	w_1	w_2	w_3	b	check
w_1	4	3	1	0	0	480	488
w_2	6	5	0	1	0	480	492
w_3	1	1	0	0	1	240	243
f	–3	–2	0	0	0	0	–5

basis	x_1	x_2	w_1	w_2	w_3	b	check
w_1	4	3	1	0	0	480	488
w_2	1	5/6	0	1/6	0	80	82
w_3	1	1	0	0	1	240	243
f	-3	-2	0	0	0	0	-5

basis	x_1	x_2	w_1	w_2	w_3	b	check
w_1	0	−2/6	1	−4/6	0	160	160
w_2	1	5/6	0	1/6	0	80	82
w_3	0	1/6	0	−1/6	1	160	161
f	0	3/6	0	3/6	0	240	241

We have

$$f - \frac{1}{2}x_2 - \frac{1}{2}w_2 = 240 \quad \Rightarrow \quad f = 240 + \frac{1}{2}x_2 + \frac{1}{2}w_2$$

Obviously, the maximum profit occurs when we set $x_2 = w_2 = 0$ and $x_1 = 80$. The maximum profit is $240 for daily by producing 80 sets of drive D_1 and 0 set of drive D_2.