FREQUENTLY ASKED QUESTIONS ON COMPLEX ANALYSIS

PREPARED BY PROFESSOR BEN M. CHEN
Department of Electrical and Computer Engineering
National University of Singapore
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1. *What are the Cauchy-Rieman equations for?*

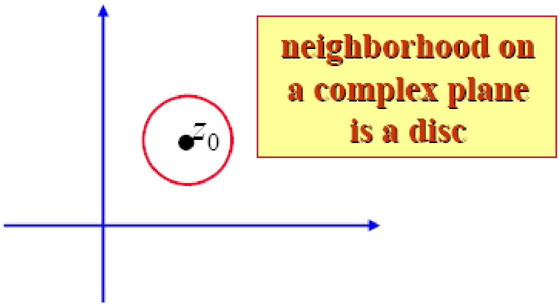
   Given a complex function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, Cauchy-Rieman equations or conditions are to force the complex derivative of the function to be identical in every direction on the complex plane. The equations or conditions are the result of forcing the partial derivatives along the x-direction and y-direction to be the same.

2. *Why do we have to force the complex derivative of a given function to be identical in every direction on the complex plane?*

   The complex derivative is with respect to $z = x + iy$. It is a 2D differentiation.

3. *If the Cauchy-Rieman equations or conditions are satisfied for a given function at a point, say z_0, can you conclude that the function is analytic at z_0?*

   No. A function $f(z)$ is said to be analytic at a point z_0 if it is analytic in a neighborhood of z_0.



4. If $f(z)$ is analytic at z_0, will the points at its neighborhood, say point A for instance, also be analytic?

   It depends. The fact that $f(z)$ is analytic at z_0 implies that the function is analytic in the neighborhood of z_0 (see Q.3 above). However, the radius of such a neighborhood might be infinitesimal. Thus, any fixed point A might not necessarily analytic.

5. *Since almost all the formula we use requires $f(z)$ to be analytic in its domain, do we need to prove $f(z)$ is analytic every time we apply the formula?*

Yes, it is indeed that in many formulae we have in the lectures, which require the function of interest, e.g., $f(z)$, to be analytic. We do need to justify that $f(z)$ is analytic when required. In many situations, however, we can prove by the following observations:

1. The sum or product of analytic functions is analytic;
2. All polynomials are analytic;
3. A rational function is analytic, except at zeroes of the denominator;
4. An analytic function of an analytic function is analytic;
5. Functions e^z, $\sin z$, $\cos z$, $\sinh z$, $\cosh z$ are analytic everywhere.

Side Note: Formulae that require certain functions integrated being analytical include the Cauchy's Integral Theorem, i.e.,

$$\oint_C f(z)dz = 0$$

which requires $f(z)$ being analytic on and inside *C*, and also Cauchy's integral formula, i.e.,

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz = 2\pi i \frac{f^{(n)}(z_0)}{n!}$$

which requires $f(z)$ being analytic on and inside *C* as well. However, note that in the above formula, the overall function integrated, i.e.,

$$\frac{f(z)}{(z-z_0)^{n+1}}$$

is not analytic at $z = z_0$ for $n \geq 0$.

On the other hand, the following formula

$$\oint_C f(z)dz = 2\pi i \,\text{Res}(f, z_0) \qquad \text{or} \qquad \oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f, z_k)$$

does not require $f(z)$ being analytic. Actually, we are more interested in the situation that $f(z)$ is not analytic for certain points (singular points) inside *C*.

6. *To prove that $f(z)$ is analytic, is there other method besides Cauchy-Rieman equations?*

Yes, another approach other than the direct utilization of the Cauchy-Rieman equations for proving a function being analytic is to find all its singular points in the region of interest. Remember that

A function is either analytic or singular at any given point.

If we can show that the function has no singular points in the region of interest. We can conclude that it is analytic inside the region.

As mentioned in previous question, in many situations, we are more interested in focusing on the singular points of the function. In fact, it is the singularity that makes complex analysis an interesting subject.

7. *To get the Laurent series expansion of $e^{1/z}$, it seems we just need to replace z with 1/z in the Taylor series expansion of e^z. Why is it valid to do such a substitution directly? Can we use such an approach for the expansion of other functions?*

Such an approach is valid because the Taylor series expansion or Laurent series expansion of a function, if existent, is *unique*. Hence, no matter what methods you use to derive the Taylor series or Laurent series expansion of a given function, all the answers will turn out to be the same. You can use such an approach in finding the series expansion of any function. Here is an example on finding the Taylor series for $f(z)=1/(z^2+z)$ on $|z-1|<1$:

$$f(z)=\frac{1}{z^2+z}=\frac{1}{z(z+1)}=\frac{1}{z}-\frac{1}{1+z}=\frac{1}{1-(1-z)}-\frac{1/2}{1-\frac{1-z}{2}}$$

$$=\left[1+(1-z)+(1-z)^2+(1-z)^3+\cdots\right]-\frac{1}{2}\left[1+\frac{1-z}{2}+\left(\frac{1-z}{2}\right)^2+\left(\frac{1-z}{2}\right)^3+\cdots\right]$$

$$=\left[1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right]-\frac{1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2-\left(\frac{z-1}{2}\right)^3+\cdots\right]$$

$$=\left(1-\frac{1}{2}\right)-\left(1-\frac{1}{2^2}\right)(z-1)+\left(1-\frac{1}{2^3}\right)(z-1)^2-\left(1-\frac{1}{2^4}\right)(z-1)^3+\cdots$$

We note that the first series converges on $|z-1|<1$ and the second series converges on $|(1-z)/2|<1$, which is equivalent to $|z-1|<2$. In overall, we can only guarantee the sum of these two series converges on $|z-1|<1$.