# Tutorial 2.1

**Question 2.1.0:** (open discussion – no solution is to be provided)**:** *What is a curve? What is the length of a curve? How to characterize a circle and a straight line? What is a complex integral? What is the upper bound of a complex integral?*

**Question 2.1.1:** Find the parametric representation and the length of the curve: $y = x^3, \quad 0 \le x \le 1$.

**Question 2.1.2:** Let $C$ be a circle with radius $r$ and centred at the origin, i.e., $C: \ z(t) = a + re^{it}$, , $t \in [0, 2\pi]$ and $f(z) = z^2$. Calculate the integral $\int_C z^2 \, dz$.

**Question 2.1.3:** Find the upper bound for the absolute value of $\int_C z \, dz$ where $C$ is the half-circle, i.e., $z(t) = e^{it}, \quad t \in [0, \pi]$, as shown in Figure 1.

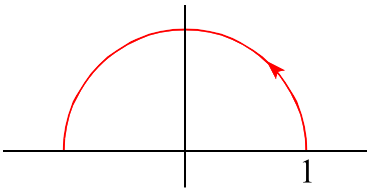


**Figure 1:** $z(t) = e^{it}, \qquad t \in [0, \pi]$

**Question 2.1.4:** Calculate the integral $\oint_{|z|=2} \frac{2z-1}{z^2 - z} dz$. Hint: You can try Cauchy's integral theorem here and note the poles inside. Draw a graph of the circle and indicate the poles.

**Question 2.1.5:** Calculate $\oint_{|z|=2} \frac{z^2+1}{z^2-1} dz$.

**Question 2.1.6:** Calculate $\oint_{|z|=2} \frac{1}{z^3(z+4)} dz$.

# Tutorial 2.2

**Question 2.2.0:** (open discussion – no solution is to be provided)**:** *What is an analytic function? What is the Cauchy-Rieman condition for? What is the derivative of a complex function? What is a singular point? What is the order of singularities? What is the Cauchy's integral theorem? What is the Cauchy's integral formula?*

**Question 2.2.1:** Find the singularities of $f(z) = \dfrac{e^z - \sin z - 1}{z^2}$.

**Question 2.2.2:** Calculate $\oint_{|z|=2} \dfrac{e^z}{z^2 - 1}\, dz$ .

**Question 2.2.3:** Calculate $\oint_{|z|=1} \dfrac{z^2 + 1}{e^z \sin z}\, dz$ .

**Question 2.2.4:** Calculate $\oint_{|z|=2} \dfrac{z}{z^2 + 2z + 2}\, dz$ .

**Question 2.2.5:** Calculate $\int_0^{2\pi} \dfrac{\sin^2 \theta}{5 + 4\cos\theta}\, d\theta$ .

# Tutorial 2.3

**Question 2.3.0:** (open discussion – no solution is to be provided)**:** *What are the Taylor series expansion and Laurent series expansion of complex functions? What are the residues of complex functions? What is Jordan's Lemma? What is argument principle? What is Rouche's Theorem?*

**Question 2.3.1:** Calculate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx$ .

**Question 2.3.2:** Calculate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\,dx$ .

**Question 2.3.3:** Calculate $\int_{-\infty}^{\infty} \frac{x\sin \pi x}{x^2+2x+5}\,dx$ .

**Question 2.3.4:** Calculate $\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+x-2}\,dx$ .

**Question 2.3.5:** From the definition of the inverse Laplace transform

$$f(t) = L^{-1}\left[F(s)\right] = \frac{1}{2\pi i}\oint_C e^{st}\,F(s)\,ds\,,$$

calculate the inverse transform of $F(s) = \frac{s}{s^2-3}$ if existent.

**Question 2.3.6:** Calculate $\oint_C \frac{1}{\cos \pi z \sin \pi z}\,dz$ , $C: |z| = \pi$ .

**Question 2.3.7:** Use Rouche's theorem determine the number of roots (zeros) of $p(z) = z^4 - 5z + 1$ that lie within the annular region $1 < |z| < 2$ .

# Tutorial 2.4

**Question 2.4.0:** (open discussion – no solution is to be provided)**:** *What is the Cauchy's steepest descent method for? What is the key idea in Cauchy's steepest descent method? What is the key idea in the Simplex Method? What are the basic operations in the Simplex Method?*

**Question 2.4.1:** Use the Cauchy's method of steepest descent to derive an iteration scheme that yields an optimal solution corresponding to the minimum value of

$$f(\mathbf{x}) = x_1^2 + 2x_1x_2 + 2x_2^2 + 3x_3^2$$

Show detailed iterations from a starting point $\mathbf{x}_0 = (\,1 \quad 2 \quad 3\,)'$ and a fixed step size $t^* = 0.1$. Calculate the results with fixed step sizes $t^* = 0.2$ and $t^* = 0.4$, respectively.

**Question 2.4.2:** Maximize $f = 11x_1 + 15x_2$ subject to the constraints

$$\begin{gathered} 3x_1 + 5x_2 \le 130 \\ -4x_1 + 5x_2 \ge 25 \\ x_1 + 5x_2 \ge 75 \\ x_1, x_2 \ge 0\,. \end{gathered}$$

**Question 2.4.3:** Minimize $f = -3x_1 + 4x_2$ subject to the constraints

$$\begin{gathered} x_1 + 3x_2 \le 54 \\ 3x_1 + x_2 \le 34 \\ -x_1 + 2x_2 \ge 12 \\ x_1, x_2 \ge 0\,. \end{gathered}$$

# Solutions to Tutorial 2.1

**Question 2.1.0:** (open discussion – no solution is to be provided)**:** *What is a curve? What is the length of a curve? How to characterize a circle and a straight line? What is a complex integral? What is the upper bound of a complex integral?*

## ***Discussion:***

1. *What is a curve?*
2. *What is the length of a curve?*
3. *How to characterize a circle and a straight line?*
4. *What is a complex integral?*
5. *What is the upper bound of a complex integral?*

**Question 2.1.1**

Find the parametric representation and the length of the curve:

$$y = x^3, \quad 0 \le x \le 1 .$$

**Solution:**

Let $x = t, \quad 0 \le t \le 1$.

$\Rightarrow \quad y = t^3$

$\Rightarrow \quad z(t) = t + it^3, \quad 0 \le t \le 1$

$$L = \int_0^1 \left|1 + i3t^2\right| dt = \int_0^1 \sqrt{1 + 9t^4}\ dt$$

A numerical solution for *L* can be found.

**Question 2.1.2**

Let *C* be a circle with radius *r* and centred at the origin, while $f(z) = z^2$.

Thus $C:\ z(t) = a + re^{it}\ ,\quad t \in [0, 2\pi]$

Calculate the integral $\int_C z^2\, dz$.

**Solution:**

By observation, the parametric representation of *C* can be rewritten as $z(t) = re^{it}$, since it is centered at the origin. Then $z'(t) = ire^{it}$ and

$$\int_C z^2\, dz = \int_0^{2\pi} ir^3 e^{i3t}\, dt = \frac{ir^3}{i3}\left[e^{i3t}\right]_0^{2\pi} = 0$$

**Question 2.1.3**

Find the upper bound for the absolute value of $\int_C z\,dz$ where $C$ is the half-circle

$C:\ z(t) = e^{it}\ , \quad t \in [0, \pi]\ .$

**Solution:**

To find the upper bound, we need to find the maximum magnitude of $f(z) = z$ on the curve $C$ and the length of curve $C$:

$$L = \int_0^{\pi} \left| e^{it} \right| dt = \int_0^{\pi} 1\,dt = \pi$$

$$\begin{aligned} M &= \max \left| e^{it} \right|, \quad t \in [0, 2\pi] \\ &= 1 \end{aligned}$$

$$\text{Thus} \quad \left| \int_C z\,dz \right| \le M\,L = \pi$$

Note that if we evaluate the integral directly, we obtain $\int_C z\,dz = \int_0^{\pi} e^{it}.ie^{it}\,dt = \int_0^{\pi} ie^{i2t}\,dt = 0$ .

**Question 2.1.4**

Calculate the integral $\oint_{|z|=2} \frac{2z-1}{z^2-z}\,dz$.

**Solution:**

Note that $\frac{2z-1}{z^2-z} = \frac{2z-1}{z(z-1)}$ has two poles 0 and 1 and these are both inside the integration path, a circle with radius 2 and centred at the origin. (see Fig. 1).

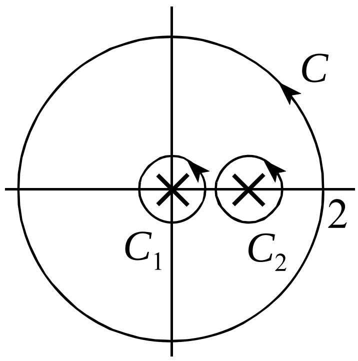


Fig. 1

Thus, we can substitute the original integral by the individual integrals along the path around the two poles as shown in Fig. 1, where $C_1$ is a circle around point $z$=0 and $C_2$ is a circle around point $z$=1:

$$\oint_{|z|=2} \frac{2z-1}{z^2-z}\,dz = \oint_{C_1} \frac{2z-1}{z^2-z}\,dz + \oint_{C_2} \frac{2z-1}{z^2-z}\,dz$$

Using partial fractions, we obtain:

$$\begin{aligned}\oint_{C_1} \frac{2z-1}{z^2-z}\,dz + \oint_{C_2} \frac{2z-1}{z^2-z}\,dz &= \oint_{C_1} \left(\frac{1}{z} + \frac{1}{z-1}\right)dz + \oint_{C_2} \left(\frac{1}{z} + \frac{1}{z-1}\right)dz \\ &= \oint_{C_1} \frac{1}{z}\,dz + \oint_{C_1} \frac{1}{z-1}\,dz + \oint_{C_2} \frac{1}{z}\,dz + \oint_{C_2} \frac{1}{z-1}\,dz\end{aligned}$$

From the Cauchy Integral Theorem, if the path $C$ encloses the point $z_0$, then

$$\oint_C (z-z_0)^n\,dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

and if a function $f(z)$ is analytic inside the region enclosed by $C$, we have

$$\oint_C f(z)\,dz = 0 .$$

Thus

$$\oint_{C_1} \frac{1}{z}\,dz = 2\pi i\,, \quad \oint_{C_1} \frac{1}{z-1}\,dz = 0\,, \quad \oint_{C_2} \frac{1}{z}\,dz = 0\,, \quad \oint_{C_2} \frac{1}{z-1}\,dz = 2\pi i$$

Hence,

$$\oint_{|z|=2} \frac{2z-1}{z^2-z}\,dz = 4\pi i$$

**Question 2.1.5**

Calculate $\oint_{|z|=2} \frac{z^2+1}{z^2-1}\, dz$ .

**Solution:**

Cauchy's integral formula states that for an analytic function $f(z)$ and $C$ a closed curve which encloses $z_0$:

$$\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)$$

Refer to Fig. 2. Note that $\frac{z^2+1}{z+1}$ is analytic inside $C_2$ and $\frac{z^2+1}{z-1}$ is analytic inside $C_1$.

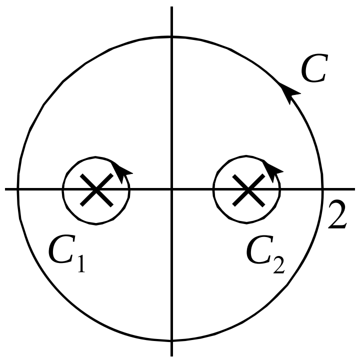


Fig. 2

$$\begin{aligned}
\oint_{|z|=2} \frac{z^2+1}{z^2-1}\, dz &= \oint_{|z|=2} \frac{z^2+1}{(z-1)(z+1)}\, dz \\
&= \oint_{C_2} \frac{\frac{z^2+1}{(z+1)}}{(z-1)}\, dz + \oint_{C_1} \frac{\frac{z^2+1}{(z-1)}}{(z+1)}\, dz \\
&= 2\pi i \left[\frac{z^2+1}{(z+1)}\right]_{z=1} + 2\pi i \left[\frac{z^2+1}{(z-1)}\right]_{z=-1} \\
&= 2\pi i - 2\pi i \\
&= 0
\end{aligned}$$

**Question 2.1.6**

Calculate $\oint_{|z|=2} \frac{1}{z^3(z+4)}\, dz$

**Solution:**

We will use the relation associated the power series, stating that

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = 2\pi i\, \frac{f^{(n)}(z_0)}{n!},$$

to solve this problem.

Refer to Fig. 3. We then find that

$$\begin{aligned}\oint_{|z|=2} \frac{1}{z^3(z+4)}\, dz &= \oint_{|z|=2} \frac{\frac{1}{z+4}}{z^3}\, dz = 0 \\ &= \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left[\frac{1}{z+4}\right]_{z=0} \\ &= \pi i \left[\frac{2}{(z+4)^3}\right]_{z=0} \\ &= \frac{2\pi i}{64}\end{aligned}$$

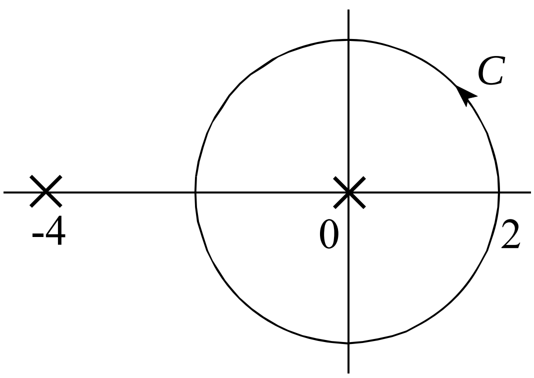


Fig. 3

# Solutions to Tutorial 2.2

**Question 2.2.0:** (open discussion – no solution is to be provided)**:** *What is an analytic function? What is the Cauchy-Rieman condition for? What is the derivative of a complex function? What is a singular point? What is the order of singularities? What is the Cauchy's integral theorem? What is the Cauchy's integral formula?*

## ***Discussion:***

1. *What is an analytic function?*
2. *What is the Cauchy-Rieman condition for?*
3. *What is the derivative of a complex function?*
4. *What is a singular point?*
5. *What is the order of singularities?*
6. *What is the Cauchy's integral theorem?*
7. *What is the Cauchy's integral formula?*

**Question 2.2.1**

Find the singularities of $f(z) = \dfrac{e^z - \sin z - 1}{z^2}$.

**Solution:**

$e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dots$

$\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dots$

Hence

$$
\begin{aligned}
f(z) &= \frac{e^z - \sin z - 1}{z^2} \\
&= \frac{1}{z^2}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right) - 1\right] \\
&= \frac{1}{z^2}\left[\frac{z^2}{2!} + 2\frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^6}{6!} + 2\frac{z^7}{7!} \dots\right] \\
&= \frac{1}{2!} + \frac{2z}{3!} + \frac{z^2}{4!} + \frac{z^4}{6!} + 2\frac{z^5}{7!} \dots
\end{aligned}
$$

The Laurent expansion of $f(z)$ has no negative powers of $z$. The function $f(z)$ therefore has a removable singularity at $z = 0$.

**Question 2.2.2**

Calculate $\oint_{|z|=2} \frac{e^z}{z^2-1}\, dz$ .

**Solution:**

By observation, $\frac{e^z}{z^2-1}$ has two simple poles at $z = \pm 1$ and both of them are inside the integration path (See Fig. 1).

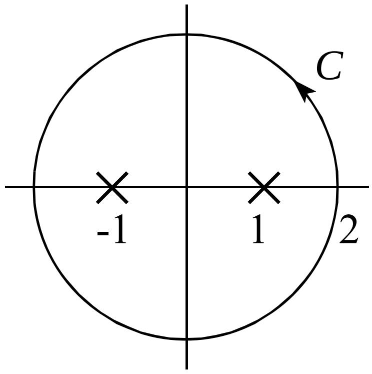


Fig. 1

According to $\oint_C f(z)\, dz = 2\pi i \sum_{i=1}^{n} \text{Res}(f, z_i)$, we have

$$\oint_{|z|=2} \frac{e^z}{z^2-1}\, dz = 2\pi i \left[\text{Res}(f,1) + \text{Res}(f,-1)\right]$$

From $\text{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0)\, f(z)$,

$$\text{Res}(f,1) = \lim_{z\to 1} \frac{e^z}{z+1} = \frac{e}{2} \qquad \text{Res}(f,-1) = \lim_{z\to -1} \frac{e^z}{z-1} = \frac{e^{-1}}{-2}$$

Hence,

$$\oint_{|z|=2} \frac{e^z}{z^2-1}\, dz = 2\pi i \left[\frac{e}{2} - \frac{e^{-1}}{2}\right]$$

$$= \pi i [e - 1/e]$$

**Question 2.2.3**

Calculate $\oint\limits_{|z|=1} \frac{z^2+1}{e^z \sin z}\,dz$ .

**Solution:**

Let $g(z) = e^z \sin z$ .

Then $g'(z) = e^z \cos z + e^z \sin z \qquad g'(0) = 1 \neq 0$ .

Therefore $g(z)$ has a 1st order zero in $z = 0$, and $f(z)$ has a simple pole in $z = 0$ .

Therefore, using the 3rd formula on p. 1-18 of the lecture notes, we find that

$$\oint\limits_{|z|=1} \frac{z^2+1}{e^z \sin z}\,dz = 2\pi i \,\mathrm{Res}\,(f,0)$$

$$= 2\pi i \left[\frac{z^2+1}{e^z \sin z + e^z \cos z}\right]_{z=0}$$

$$= 2\pi i$$

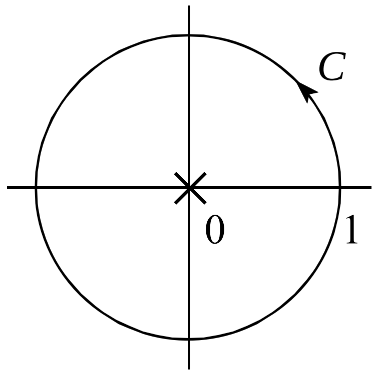


Fig. 2

**Question 2.2.4**

Calculate $\oint_{|z|=2} \frac{z}{z^2+2z+2}\,dz$ .

**Solution:**

$f(z)=\frac{z}{z^2+2z+2}$ has simple poles at $z=-1\pm i$ . See Fig. 3.

$$\mathrm{Res}(f,-1+i)=\left[\frac{z}{z-(-1-i)}\right]_{z=-1+i}=\tfrac{1}{2}(1+i)$$

$$\mathrm{Res}(f,-1-i)=\left[\frac{z}{z-(-1+i)}\right]_{z=-1-i}=\tfrac{1}{2}(1-i)$$

$$\oint_{|z|=2} \frac{z}{z^2+2z+2}\,dz=2\pi i\left[\mathrm{Res}(f,-1+i)+\mathrm{Res}(f,-1-i)\right]$$

$$=2\pi i$$

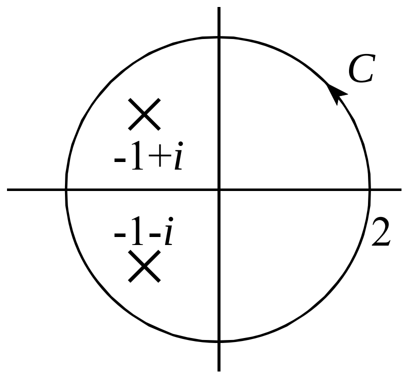


Fig. 3

**Question 2.2.5**

Calculate $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta$ .

**Solution:**

From $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \oint_{|z|=1} f\left[\tfrac{1}{2}\left(z+\tfrac{1}{z}\right), \tfrac{1}{2i}\left(z-\tfrac{1}{z}\right)\right] \frac{1}{iz}\, dz$ , we have

$$\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta = \oint_{|z|=1} \frac{\left[\tfrac{1}{2i}\left(z-\tfrac{1}{z}\right)\right]^2}{5+4\left[\tfrac{1}{2}\left(z+\tfrac{1}{z}\right)\right]} \frac{1}{iz}\, dz$$

$$= -\frac{1}{4i} \oint_{|z|=1} \frac{z^4-2z^2+1}{z^2(2z^2+5z+2)}\, dz$$

$$= -\frac{1}{4i} \oint_{|z|=1} \frac{(z^2-1)^2}{2z^2(z+\tfrac{1}{2})(z+2)}\, dz$$

$$= -\frac{1}{4i}\; 2\pi i \left[\operatorname{Res}(f,0) + \operatorname{Res}(f,-\tfrac{1}{2})\right]$$

$$\operatorname{Res}(f,0) = \frac{1}{1!} \lim_{z\to 0} \frac{d}{dz}\left[\frac{(z^2-1)^2}{2(z+\tfrac{1}{2})(z+2)}\right]$$

$$= \lim_{z\to 0} \frac{2(z^2-1)\,2z\,(2z^2+5z+2) - (z^2-1)^2(4z+5)}{(2z^2+5z+2)^2}$$

$$= -\tfrac{5}{4}$$

$$\operatorname{Res}(f,-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}} \frac{(z^2-1)^2}{2z^2(z+2)}$$

$$= \tfrac{3}{4}$$

Note that pole $z=-2$ falls outside of the region enclosed by the integral path. See Fig. 4.

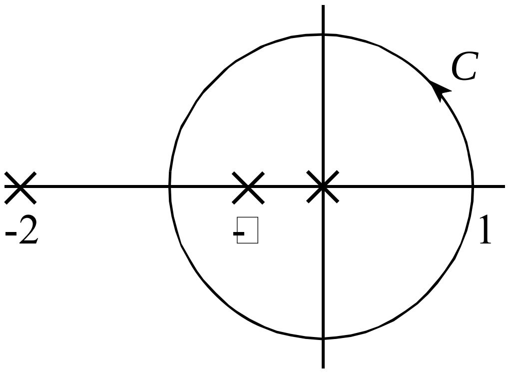


Fig. 4

Thus $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\, d\theta = -\frac{1}{4i} \,.\, 2\pi i \left(\tfrac{3}{4} - \tfrac{5}{4}\right) = \frac{\pi}{4}$ .

# Solutions to Tutorial 2.3

**Question 2.3.0:** (open discussion – no solution is to be provided)**:** *What are the Taylor series expansion and Laurent series expansion of complex functions? What are the residues of complex functions? What is Jordan's Lemma? What is argument principle? What is Rouche's Theorem?*

***Discussion:***

1. *What is the Taylor series expansion of complex functions?*
2. *What is the Laurent series expansion of complex functions?*
3. *What are the residues of complex functions?*
4. *What is Jordan's Lemma?*
5. *What is argument principle?*
6. *What is Rouche's Theorem?*

**Question 2.3.1**

Calculate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx$.

**Solution:** Let

$$f(z) = \frac{1}{(z^2+1)(z^2+9)} = \frac{1}{(z-i)(z+i)(z-3i)(z+3i)}$$

From Fig. 1, only $z = i$ and $z = 3i$ are in the upper half plane.

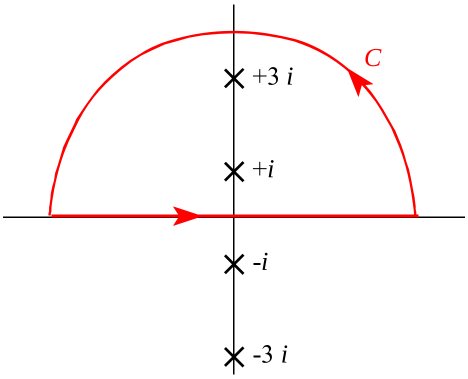


Fig. 1

Thus,

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx = 2\pi i \left[\text{Res}(f,i) + \text{Res}(f,3i)\right]$$

$$= 2\pi i \left\{ \left[\frac{1}{(z+i)(z-3i)(z+3i)}\right]_{z=i} + \left[\frac{1}{(z-i)(z+i)(z+3i)}\right]_{z=3i} \right\}$$

$$= 2\pi i \left[\frac{-i}{16} + \frac{i}{48}\right] = \frac{\pi}{12}$$

**Question 2.3.2**

Calculate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx$ .

**Solution:**

Let $f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}$ . The function has a 2nd order pole at $z = i$ . See Fig. 2.

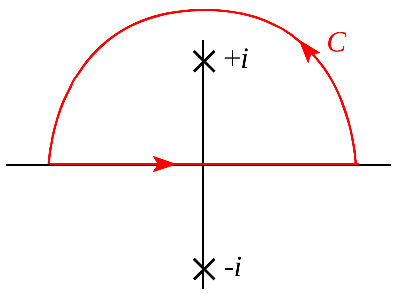


Fig. 2

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2}\, dx = 2\pi i \,\mathrm{Res}\,(f, i) = 2\pi i \frac{1}{1!} \lim_{z\to i} \frac{d}{dz}\left[\frac{1}{(z+i)^2}\right]$$

$$= 2\pi i \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{\pi}{2}$$

**Question 2.3.3**

Calculate $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx$ .

**Solution:**

Let $f(z) = \frac{z e^{i\pi z}}{z^2 + 2z + 5} = \frac{z e^{i\pi z}}{(z - (-1 + 2i))(z - (-1 - 2i))}$ . The function has a pole in the upper half-plane at $z = -1 + 2i$ .

Thus

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5} dx &= \mathrm{Im}\left[ 2\pi i \,\mathrm{Res}\left( \frac{z e^{i\pi z}}{z^2 + 2z + 5}, -1 + i2 \right) \right] \\
&= \mathrm{Im}\left[ 2\pi i \left( \frac{z e^{i\pi z}}{z - (-1 - 2i)} \right)_{z=-1+i2} \right] = \mathrm{Im}\left[ 2\pi i \left( \frac{(-1 + i2) e^{-i\pi - 2\pi}}{4i} \right) \right] \\
&= \mathrm{Im}\left[ \frac{\pi}{2}(-1 + i2)(-e^{-2\pi}) \right] = \mathrm{Im}\left[ e^{-2\pi} \left( \frac{\pi}{2} - i\pi \right) \right] = -\pi e^{-2\pi}
\end{aligned}
$$

**Question 2.3.4**

Calculate $\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+x-2}\,dx$.

**Solution:**

Refer to Fig. 3. Consider

$$\oint_C \frac{z e^{iz}}{(z-1)(z+2)}dz = \left[\int_{-R}^{-2-\rho} + \int_{C_1(\rho)} + \int_{-2+\rho}^{1-\rho} + \int_{C_2(\rho)} + \int_{1+\rho}^{R} + \int_{C_3}\right]\frac{z e^{iz}}{(z-1)(z+2)}dz = 0$$

Let $f(z) = \frac{z e^{iz}}{(z-1)(z+2)}$, then

$$\lim_{\rho\to 0}\int_{C_1(\rho)} f(z)\,dz = -\pi i \,\text{Res}(f,-2) = -\pi i \lim_{z\to -2}\frac{z e^{iz}}{z-1} = -\frac{2\pi i e^{-2i}}{3}$$

$$\lim_{\rho\to 0}\int_{C_2(\rho)} f(z)\,dz = -\pi i \,\text{Res}(f,1) = -\pi i \lim_{z\to 1}\frac{z e^{iz}}{z+2} = -\frac{\pi i e^{i}}{3}$$

Thus $\int_{-\infty}^{\infty} f(z)\,dz - \frac{2\pi i e^{-2i}}{3} - \frac{\pi i e^{i}}{3} = 0$

$$\Rightarrow \int_{-\infty}^{\infty} f(z)\,dz = \frac{2\pi i e^{-2i}}{3} + \frac{\pi i e^{i}}{3}$$

$$= \frac{2\pi i}{3}(\cos 2 - i\sin 2) + \frac{\pi i}{3}(\cos 1 + i\sin 1)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\cos x}{x^2+x-2}dx = \text{Re}\left(\frac{2\pi i}{3}(\cos 2 - i\sin 2) + \frac{\pi i}{3}(\cos 1 + i\sin 1)\right)$$

$$= \frac{2\pi}{3}\sin 2 - \frac{\pi}{3}\sin 1$$

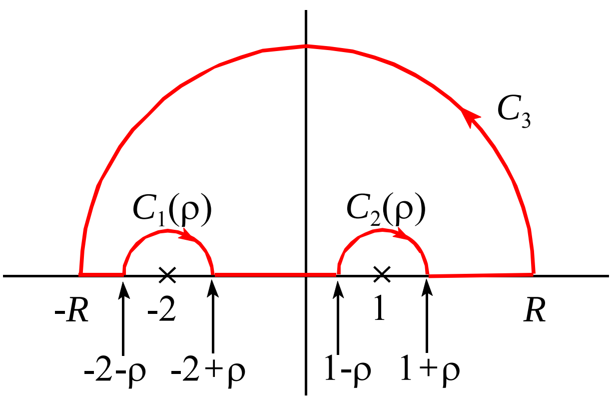


Fig. 3

**Question 2.3.5**

From the definition of the inverse Laplace transform

$$f(t) = L^{-1}\left[F(s)\right] = \frac{1}{2\pi i}\oint_C e^{st}\, F(s)\, ds\,,$$

calculate the inverse transforms of $F(s) = \dfrac{s}{s^2-3}$ if existent.

**Solution:** It is straightforward to verify that for the given $F(s)$,

$$\lim_{s\to\infty}\frac{s}{s^2-3} = 0 \quad \text{and} \quad \lim_{s\to\infty} s\cdot\frac{s}{s^2-3} = 1$$

Thus, the inverse Laplace transform of $F(s)$ exists and is given by

$$\begin{aligned} f(t) &= \operatorname{Res}\left[\frac{z\,e^{zt}}{z^2-3}, \sqrt{3}\right] + \operatorname{Res}\left[\frac{z\,e^{zt}}{z^2-3}, -\sqrt{3}\right] \\ &= \left[\frac{z\,e^{zt}}{z+\sqrt{3}}\right]_{z=\sqrt{3}} + \left[\frac{z\,e^{zt}}{z-\sqrt{3}}\right]_{z=-\sqrt{3}} \\ &= \frac{\sqrt{3}\,e^{\sqrt{3}t}}{2\sqrt{3}} + \frac{-\sqrt{3}\,e^{-\sqrt{3}t}}{-2\sqrt{3}} \\ &= \frac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2} = \cosh\sqrt{3}t \end{aligned}$$

**Question 2.3.6**

Calculate $\oint_C \frac{1}{\cos \pi z \sin \pi z} dz$ , $C : |z| = \pi$ .

**Solution:**

$$\oint_C \frac{1}{\cos \pi z \sin \pi z} dz = \frac{1}{\pi} \oint_C \frac{\pi}{\cos^2 \pi z} \frac{1}{\sin \pi z / \cos \pi z} dz$$

$$= \frac{1}{\pi} \oint_C \frac{\pi \sec^2 \pi z}{\tan \pi z} dz$$

$$= \frac{1}{\pi} \oint_C \frac{f'(z)}{f(z)} dz \qquad , \quad f(z) = \tan \pi z$$

We can therefore apply the argument theorem to solve the problem. Since

$$\tan \pi z = \frac{\sin \pi z}{\cos \pi z} ,$$

the zeros of $f(z)$ are the zeros of $\sin \pi z$ , which are $z = 0, \pm 1, \pm 2, \pm 3, \pm 4 \ldots$ Of these, only $z = 0, \pm 1, \pm 2, \pm 3$ (seven of them) lie within *C*. Similarly, the poles of $f(z)$ are the zeros of $\cos \pi z$ , which are $z = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm \frac{7}{2} \ldots$ Of these, only $z = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}$ (6 of them) lie within *C*. Thus by the argument theorem

$$\oint_C \frac{1}{\cos \pi z \sin \pi z} dz = \frac{1}{\pi} \oint_C \frac{f'(z)}{f(z)} dz = \frac{1}{\pi} 2\pi i [7 - 6] = 2i .$$

**Question 2.3.7**

Use Rouche's theorem determine the number of roots (zeros) of $p(z) = z^4 - 5z + 1$ that lie within the annular region $1 < |z| < 2$.

**Solution:**

$C_1 : |z| = 1$ Choose $f(z) = -5z$ and $g(z) = z^4 + 1$.

on $C_1$: $|f(z)| = 5 > 2 = |g(z)|$.

By Rouche's theorem $p(z) = f(z) + g(z)$ has one zero inside $C_1$, since $f(z)$ has one zero inside $C_1$.

$C_2 : |z| = 2$ Choose $f(z) = z^4$ and $g(z) = -5z + 1$.

On $C_2$: $|f(z)| = 16 > 11 = |g(z)|$.

Thus by Rouche's theorem, $p(z) = f(z) + g(z)$ has four zeros inside $C_2$, since $f(z)$ has four zeros inside $C_2$.

Hence there are 4−1=3 zeros of $p(z)$ inside the annular region $1 < |z| < 2$.

# Solutions to Tutorial 2.4

**Question 2.4.0:** (open discussion – no solution is to be provided)**:** *What is the Cauchy's steepest descent method for? What is the key idea in Cauchy's steepest descent method? What is the key idea in the Simplex Method? What are the basic operations in the Simplex Method?*

## ***Discussion:***

1. *What is the Cauchy's steepest descent method for?*
2. *What is the key idea in Cauchy's steepest descent method?*
3. *What is the key idea in the Simplex Method?*
4. *What are the basic operations in the Simplex Method?*

**Question 2.4.1**

Use the Cauchy's method of steepest descent to derive an iteration scheme that yields an optimal solution corresponding to the minimum value of

$$f(\mathbf{x}) = x_1^2 + 2x_1x_2 + 2x_2^2 + 3x_3^2$$

Show detailed iterations from a starting point $\mathbf{x}_0 = (\,1 \quad 2 \quad 3\,)'$ and a fixed step size $t^* = 0.1$. Calculate the results with fixed step sizes $t^* = 0.2$ and $t^* = 0.4$, respectively.

**Solution:**

We first compute the gradient of the objective function, i.e.,

$$\nabla f(\mathbf{x}) = \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + 2x_2 \\ 2x_1 + 4x_2 \\ 6x_3 \end{pmatrix}$$

The resulting iteration for $t^* = 0.1$ is

$$\mathbf{x}_i(t) = \mathbf{x}_{i-1} - t^* \nabla f(\mathbf{x}_{i-1}) = \begin{pmatrix} x_{1,i-1} - 2t x_{1,i-1} - 2t x_{2,i-1} \\ x_{2,i-1} - 2t x_{1,i-1} - 4t x_{2,i-1} \\ x_{3,i-1} - 6t x_{3,i-1} \end{pmatrix}$$

| **Step $i$** | $\mathbf{x}_i$: $x_{1,i}$ | $x_{2,i}$ | $x_{3,i}$ | **Fixed Step Size $t^*$** |
|---|---|---|---|---|
| 0 | 1.0000 | 2.0000 | 3.0000 | 0.1 |
| 1 | 0.4000 | 1.0000 | 1.2000 | 0.1 |
| 2 | 0.1200 | 0.5200 | 0.4800 | 0.1 |
| 3 | –0.0080 | 0.2880 | 0.1920 | 0.1 |
| 4 | –0.0640 | 0.1744 | 0.0768 | 0.1 |
| 5 | –0.0861 | 0.1174 | 0.0307 | 0.1 |
| 6 | –0.0924 | 0.0877 | 0.0123 | 0.1 |
| 7 | –0.0914 | 0.0711 | 0.0049 | 0.1 |
| : | : | : | : | 0.1 |
| 10 | –0.0765 | 0.0488 | 0.0003 | 0.1 |
| : | : | : | : | 0.1 |
| 20 | –0.0349 | 0.0215 | 0.0000 | 0.1 |
| : | : | : | : | 0.1 |
| 70 | –0.0007 | 0.0004 | 0.0000 | **Slow** |

The result for $t^* = 0.2$:

| **Step** $i$ | $\mathbf{x}_i$ $x_{1,i}$ | $x_{2,i}$ | $x_{3,i}$ | **Fixed Step Size** $t^*$ |
|---|---|---|---|---|
| 0 | 1.0000 | 2.0000 | 3.0000 | 0.2 |
| 1 | –0.2000 | 0.0000 | –0.6000 | 0.2 |
| 2 | –0.1200 | 0.0800 | 0.1200 | 0.2 |
| 3 | –0.1040 | 0.0640 | –0.0240 | 0.2 |
| 4 | –0.0880 | 0.0544 | 0.0048 | 0.2 |
| 5 | –0.0746 | 0.0461 | –0.0010 | 0.2 |
| 6 | –0.1503 | 0.0390 | 0.0002 | 0.2 |
| 7 | –0.0535 | 0.0331 | 0.0000 | 0.2 |
| 8 | –0.0453 | 0.0280 | 0.0000 | 0.2 |
| : | : | : | : | 0.2 |
| 30 | –0.0012 | 0.0007 | 0.0000 | **Faster** |

The result for $t^* = 0.4$:

| **Step** $i$ | $\mathbf{x}_i$ $x_{1,i}$ | $x_{2,i}$ | $x_{3,i}$ | **Fixed Step Size** $t^*$ |
|---|---|---|---|---|
| 0 | 1.0000 | 2.0000 | 3.0000 | 0.4 |
| 1 | –1.4000 | –2.0000 | –4.2000 | 0.4 |
| 2 | 1.3200 | 2.3200 | 5.8800 | 0.4 |
| 3 | –1.5920 | –2.4480 | –8.2320 | 0.4 |
| 4 | 1.6400 | 2.7424 | 11.5248 | 0.4 |
| 5 | –1.8659 | –2.9574 | –16.1347 | 0.4 |
| 6 | 1.9928 | 3.2672 | 22.5886 | 0.4 |
| 7 | –2.2152 | –3.5545 | –31.6241 | 0.4 |
| 8 | 2.4006 | 3.9049 | 44.2737 | 0.4 |
| 9 | –2.6438 | –4.2634 | –61.9831 | **Failed** |

**Question 2.4.2**

Maximize $f = 11x_1 + 15x_2$ subject to the constraints

$$3x_1 + 5x_2 \le 130$$
$$-4x_1 + 5x_2 \ge 25$$
$$x_1 + 5x_2 \ge 75$$
$$x_1, x_2 \ge 0 .$$

**Solution:**

Forget about the artificial variables introduced in the textbook. We can solve this problem using the procedure suggested by Yin Mingbao in the presentation notes, i.e., by converting the constraints as

$$\left\{\begin{array}{c} 3x_1 + 5x_2 \le 130 \\ -4x_1 + 5x_2 \ge 25 \\ x_1 + 5x_2 \ge 75 \\ x_1, x_2 \ge 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} 3x_1 + 5x_2 \le 130 \\ 4x_1 - 5x_2 \le -25 \\ -x_1 - 5x_2 \le -75 \\ x_1, x_2 \ge 0 \end{array}\right\}$$

Inserting slack variables and artificial variables as required yields

$$3x_1 + 5x_2 + w_1 = 130$$
$$4x_1 - 5x_2 + w_2 = -25$$
$$-x_1 - 5x_2 + w_3 = -75$$
$$f - 11x_1 - 15x_2 = 0, \quad x_1, x_2, w_1, w_2, w_3 \ge 0$$

| Basis | $x_1$ | $x_2$ | $w_1$ | $w_2$ | $w_3$ | $b$ | check |
|---|---|---|---|---|---|---|---|
| $w_1$ | 3 | 5 | 1 | 0 | 0 | 130 | 139 |
| $w_2$ | 4 | –5 | 0 | 1 | 0 | –25 | –25 |
| $w_3$ | –1 | –5 | 0 | 0 | 1 | –75 | –80 |
| $f$ | –11 | –15 | 0 | 0 | 0 | 0 | –26 |
| $w_1$ | 3 | 5 | 1 | 0 | 0 | 130 | 139 |
| $w_2$ | –4/5 | 1 | 0 | –1/5 | 0 | 5 | 5 |
| $w_3$ | –1 | –5 | 0 | 0 | 1 | –75 | –80 |
| $f$ | –11 | –15 | 0 | 0 | 0 | 0 | –26 |

| Basis | $x_1$ | $x_2$ | $w_1$ | $w_2$ | $w_3$ | $b$ | check |
|---|---|---|---|---|---|---|---|
| $w_1$ | 7 | 0 | 1 | 1 | 0 | 105 | 114 |
| $x_2$ | –4/5 | 1 | 0 | –1/5 | 0 | 5 | 5 |
| $w_3$ | –5 | 0 | 0 | –1 | 1 | –50 | –55 |
| $f$ | –23 | 0 | 0 | –3 | 0 | 75 | 49 |
| $w_1$ | 7 | 0 | 1 | 1 | 0 | 105 | 114 |
| $x_2$ | –4/5 | 1 | 0 | –1/5 | 0 | 5 | 5 |
| $w_5$ | 1 | 0 | 0 | 1/5 | –1/5 | 10 | 11 |
| $f$ | –23 | 0 | 0 | –3 | 0 | 75 | 49 |
| $w_1$ | 0 | 0 | 1 | –2/5 | 7/5 | 35 | 37 |
| $x_2$ | 0 | 1 | 0 | –1/25 | –4/25 | 13 | 69/5 |
| $x_1$ | 1 | 0 | 0 | 1/5 | –1/5 | 10 | 11 |
| $f$ | 0 | 0 | 0 | 8/5 | –23/5 | 305 | 302 |
| $w_1$ | 0 | 0 | 5/7 | –2/7 | 1 | 25 | 185/7 |
| $x_2$ | 0 | 1 | 0 | –1/25 | –4/25 | 13 | 69/5 |
| $x_1$ | 1 | 0 | 0 | 1/5 | –1/5 | 10 | 11 |
| $f$ | 0 | 0 | 0 | 8/5 | –23/5 | 305 | 302 |

| Basis | $x_1$ | $x_2$ | $w_1$ | $w_2$ | $w_3$ | $b$ | check |
|---|---|---|---|---|---|---|---|
| $w_3$ | 0 | 0 | 5/7 | –2/7 | 1 | 25 | 185/7 |
| $x_2$ | 0 | 1 | 4/35 | –3/35 | 0 | 17 | 631/5 |
| $x_1$ | 1 | 0 | 1/7 | 1/7 | 0 | 15 | 114/7 |
| $f$ | 0 | 0 | 23/7 | 2/7 | 0 | 420 | 2965/7 |

$\therefore \quad f_{\max} = 420$ with $x_1 = 15$, $x_2 = 17$.

**Question 2.4.3**

Minimize $f = -3x_1 + 4x_2$ subject to the constraints

$$x_1 + 3x_2 \le 54$$
$$3x_1 + x_2 \le 34$$
$$-x_1 + 2x_2 \ge 12$$
$$x_1, x_2 \ge 0 .$$

**Solution:**
If *f* denotes the objective function to be minimized, we write

$$g = -f .$$

We then determine $g_{\max}$ in the normal way, and finally,

$$f_{\min} = -(g_{\max}) .$$

Inserting slack variables as required then yields

$$x_1 + 3x_2 + w_1 = 54$$
$$3x_1 + x_2 + w_2 = 34$$
$$x_1 - 2x_2 + w_3 = -12$$
$$g - 3x_1 + 4x_2 = 0 , \quad x_1, x_2, w_1, w_2, w_3 \ge 0$$

| Basis | $x_1$ | $x_2$ | $w_1$ | $w_2$ | $w_3$ | $b$ | check |
|---|---|---|---|---|---|---|---|
| $w_1$ | 1 | 3 | 1 | 0 | 0 | 54 | 59 |
| $w_2$ | 3 | 1 | 0 | 1 | 0 | 34 | 39 |
| $w_4$ | 1 | –2 | 0 | 0 | 1 | –12 | –12 |
| $g$ | –3 | 4 | 0 | 0 | 0 | 0 | 1 |
| $w_1$ | 1 | 3 | 1 | 0 | 0 | 54 | 59 |
| $w_2$ | 1 | 1/3 | 0 | 1/3 | 0 | 34/3 | 13 |
| $w_3$ | 1 | –2 | 0 | 0 | 1 | –12 | –12 |
| $g$ | –3 | 4 | 0 | 0 | 0 | 0 | 1 |
| $w_1$ | 0 | 8/3 | 1 | –1/3 | 0 | 128/3 | 46 |
| $x_1$ ~~$w_2$~~ | 1 | 1/3 | 0 | 1/3 | 0 | 34/3 | 13 |
| $w_3$ | 0 | –7/3 | 0 | –1/3 | 1 | –70/3 | –25 |
| $g$ | 0 | 5 | 0 | 1 | 0 | 34 | 40 |
| $w_1$ | 0 | 8/3 | 1 | –1/3 | 0 | 128/3 | 46 |
| $x_1$ | 1 | 1/3 | 0 | 1/3 | 0 | 34/3 | 13 |
| $w_3$ | 0 | 1 | 0 | 1/7 | –3/7 | 10 | 75/7 |
| $g$ | 0 | 5 | 0 | 1 | 0 | 34 | 40 |
| $w_1$ | 0 | 0 | 1 | –5/7 | 8/7 | 16 | 122/7 |
| $x_1$ | 1 | 0 | 0 | 2/7 | 1/7 | 8 | 66/7 |
| $x_2$ ~~$w_3$~~ | 0 | 1 | 0 | 1/7 | –3/7 | 10 | 75/7 |
| $g$ | 0 | 0 | 0 | 2/7 | 15/7 | –16 | –95/7 |

$g_{max} = -16 \quad \therefore \quad f_{min} = 16$ with $x_1 = 8$, $x_2 = 10$.