EE2012 Mid-Term Test 2 (Complex Analysis)

LT 6, 9:00–9:50AM, 15 APRIL 2011

Name: ______________________________ Matric No: ________________________ Score: [ ]

NOTE: ANSWER ALL QUESTIONS AND ANSWER DIRECTLY ON THE TEST PAPER…

Q.1 Show that the following function is analytic on the whole complex plane and find its derivative:

$$f(z) = \left[x^3 + x^2 - (3x+1)y^2\right] + i\left[xy(3x+2) - y^3\right]$$

(30 marks)

Solution: Noting that

$$u(x,y) = x^3 + x^2 - (3x+1)y^2, \quad v(x,y) = xy(3x+2) - y^3$$

$$\Downarrow$$

$$\frac{\partial u}{\partial x} = 3x^2 + 2x - 3y^2 = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = 6xy + 2y = -\frac{\partial u}{\partial y}$$

it is clear that Cauchy-Riemann equations hold for all x and y. Its derivative is given by

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \left(3x^2 + 2x - 3y^2\right) + i\left(6xy + 2y\right)$$

which is continuous. Thus, the function is analytic on the whole complex plane.

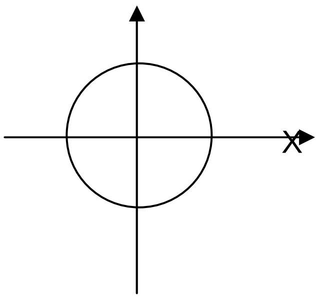
Q.2 Show that $\overline{e^{z}} = e^{\bar{z}}$, where $\overline{e^{z}}$ is the complex conjugate of e^{z} and $\bar{z}$ is the complex conjugate of z. (15 marks)

Proof. Let $z = x + i\,y$

$$e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}(\cos y + i\sin y)$$
$$\Downarrow$$
$$\overline{e^{z}} = e^{x}(\cos y - i\sin y) = e^{x}e^{-iy} = e^{x-iy} = e^{\bar{z}}$$

Q.3 Show that $\int_C \frac{1}{z-2}\,dz = 0, \quad C = \{\, z(t) = e^{it} \mid 0 \le t \le 2\pi \,\}$ (15 marks)

Proof. The given function is analytic at every point enclosed by *C*. Thus, by Cauchy's Integral Theorem, its integration is 0.



Q.4 Compute $\int_C \mathrm{Re}(z)\,dz, \quad C = \{\, z(t) = 1 + it \mid 1 \le t \le 2 \,\}$ (15 marks)

Solution: By the definition of complex integral, we have

$$\int_C \mathrm{Re}(z)\,dz = \int_1^2 \mathrm{Re}(1+it)(1+it)'dt = \int_1^2 1\cdot i\cdot dt = i\cdot t\Big|_1^2 = i$$

Q.5 Find the Taylor series expansion of $f(z)=\frac{z}{z-1}$ at $z_0=1$ and its corresponding radius of convergence. (10 marks)

Solution: The given function is not analytic at $z_0=1$. Thus, there is no Taylor series expansion for f(z) at $z_0=1$.

Q.6 Find the Taylor series expansion of $f(z)=\frac{z}{z+1}$ at $z_0=1$ and its corresponding radius of convergence. (15 marks)

Solution:

$$f(z)=\frac{z}{z+1}=\frac{z+1-1}{z+1}=1-\frac{1}{z+1}=1-\frac{1}{(z-1)+2}=1-\frac{1}{2}\cdot\frac{1}{1+\frac{z-1}{2}}$$

$$=1-\frac{1}{2}\cdot\frac{1}{1-\left(\frac{1-z}{2}\right)}=1-\frac{1}{2}\left(1+\left(\frac{1-z}{2}\right)+\left(\frac{1-z}{2}\right)^2+\left(\frac{1-z}{2}\right)^3+\cdots\right)$$

$$=\frac{1}{2}+\frac{1}{2^2}(z-1)-\frac{1}{2^3}(z-1)^2+\frac{1}{2^4}(z-1)^3+\cdots$$

Its radius of convergence is given by

$$\left|\frac{1-z}{2}\right|<1 \quad\Rightarrow\quad |z-1|<r^*=2$$