

Solutions to EE2008 Homework Assignment 2

- Q.1 Consider the circuits given in Fig. 1 below. Assume that both the capacitor and the inductor are initially discharged.

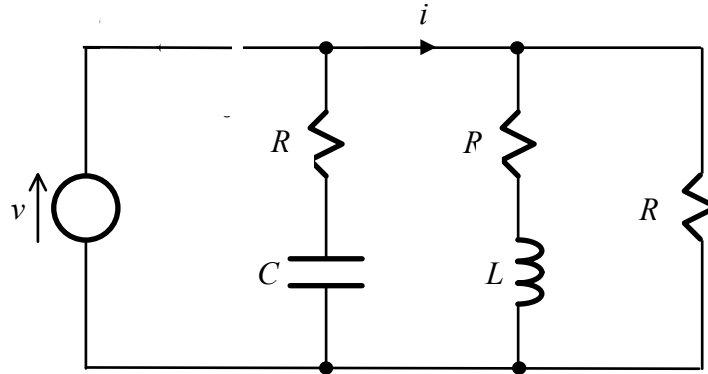


Fig. 1

- Derive the associated state and output equations with the input being the voltage source v and the output being the current i .
- Compute the transfer function of the circuit. What are the poles of the system? Is it stable? Why?
- Compute the zero response and complete response of the circuit if $v = 1$ volt.

Solution:

- Derive the associated state and output equations with the input being the voltage source v and the output being the current $i(t)$.

For capacitor,

$$i_C(t) = C \frac{dv_C(t)}{dt}$$

Applying KVL to the left loop, we have

$$i_C R + v_C = RC \frac{dv_C}{dt} + v_C = v \Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC} v_C + \frac{1}{RC} v$$

For the inductor,

$$v_L = L \frac{di_L}{dt}$$

Applying KVL,

$$v_L + i_L R = L \frac{di_L}{dt} + i_L R = v \Rightarrow \frac{di_L}{dt} = -\frac{R}{L} i_L + \frac{1}{L} v$$

Thus, we have the state equation

$$\begin{pmatrix} \dot{v}_C \\ \dot{i}_L \end{pmatrix} = \begin{bmatrix} -\frac{1}{RC} & 0 \\ 0 & -\frac{R}{L} \end{bmatrix} \begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix} v$$

The output $i = i_L + \frac{1}{R}v = i_L + \frac{1}{R}v = [0 \quad 1] \begin{pmatrix} v_C \\ i_L \end{pmatrix} + \frac{1}{R}v$

- (b) Compute the transfer function of the circuit. What are the poles of the system? Is it stable? Why?

The transfer function is then given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = [0 \quad 1] \begin{bmatrix} s + \frac{1}{RC} & 0 \\ 0 & s + \frac{R}{L} \end{bmatrix}^{-1} \begin{pmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{pmatrix} + \frac{1}{R} = \frac{\frac{1}{R}s + \frac{2}{L}}{s + \frac{R}{L}}$$

The pole of the circuit is given by

$$s = -\frac{R}{L} < 0$$

Thus, the circuit is stable. In fact, the system has another pole at $s = -\frac{1}{RC} < 0$, which

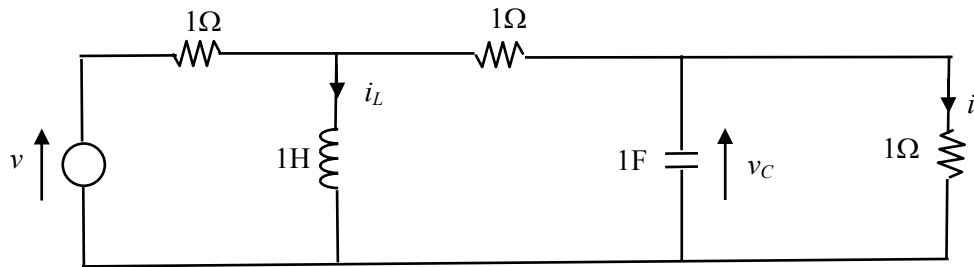
is cancelled with the zero of the system. In both cases, the system is stable.

- (c) Compute the zero response and the complete response of the circuit if $v = 1$ volt.

The zero response of the circuit is zero as the initial values of the state variables are zero. The complete response is then given by the response due to the external voltage source, i.e.,

$$i(t) = L^{-1} \left\{ \frac{\frac{1}{R}s + \frac{2}{L}}{s + \frac{R}{L}} \cdot \frac{1}{s} \right\} = L^{-1} \left\{ \frac{\frac{2}{R}}{s} - \frac{\frac{1}{R}}{s + \frac{R}{L}} \right\} = \frac{2}{R} - \frac{1}{R} e^{-(R/L)t}$$

Q.2 Consider the circuits given in Fig. 2 below. Assume that both the capacitor and the inductor are initially discharged.



- Derive the associated state and output equations with the input being the voltage source v and the output being the current i .
- Compute the transfer function of the circuit. What are the poles of the system? Is it stable? Why?
- Let $v = 1$ V. Compute the zero input response and complete response of the circuit.
- Sketch the complete response obtained in Part (c).

Solution:

- Derive the associated state and output equations with the input being the voltage source v and the output being the current i .

$$\begin{pmatrix} \dot{v}_c \\ \dot{i}_L \end{pmatrix} = \begin{bmatrix} -1.5 & -0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{pmatrix} v_c \\ i_L \end{pmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} v, \quad i = v_c = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} v_c \\ i_L \end{pmatrix} + 0 \cdot v$$

- Compute the transfer function of the circuit. What are the poles of the system? Is it stable? Why?

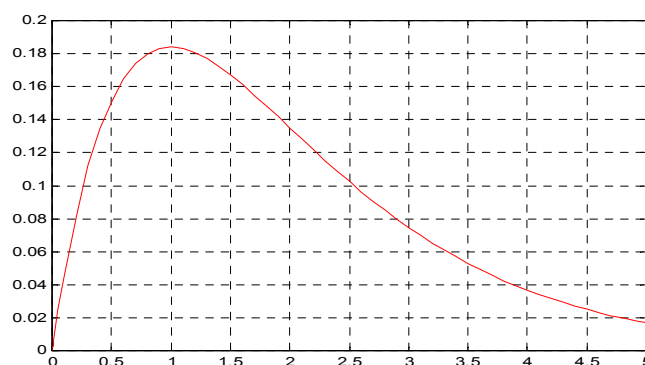
$$H(s) = \frac{0.5s}{s^2 + 2s + 1} \Rightarrow \text{poles} = -1, -1 \Rightarrow \text{stable as both poles are in LHP.}$$

- Let $v = 1$ V. Compute the zero input response and complete response of the circuit.

$$\mathbf{C}(\mathbf{sI} - \mathbf{A})^{-1} \mathbf{x}_0 = 0 \Rightarrow \text{zero input response} = 0$$

$$v(t) = L^{-1}\{H(s)U(s)\} = L^{-1}\left\{\frac{0.5s}{s^2 + 2s + 1} \cdot \frac{1}{s}\right\} = L^{-1}\left\{\frac{0.5}{(s+1)^2}\right\} = 0.5te^{-t}$$

- Sketch the complete response obtained in Part (c).



Q.3 Consider the circuit given in Fig. 3 below. Assume that the capacitor is initially charged to 1V and the inductor is initially uncharged.

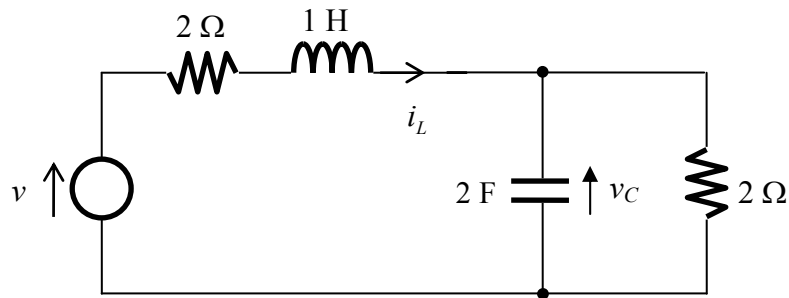


Fig. 3

- Derive the associated state and output equations with the input being the voltage source, v , and the output being the current, i_L .
- Compute the transfer function of the circuit. What are the poles of the system? Is it stable? Why?
- Compute the zero response and the complete response of the circuit if $v = 5V$.
- Sketch the complete response obtained in Part (c) for up to 5 seconds.

Solution:

$$(a) \quad \begin{pmatrix} \dot{i}_L \\ \dot{v}_C \end{pmatrix} = \begin{bmatrix} -2 & -1 \\ 1/2 & -1/4 \end{bmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v, \quad y = [1 \quad 0] \begin{pmatrix} i_L \\ v_C \end{pmatrix} + 0v$$

$$(b) \quad \mathbf{H}(s) = \frac{s + 0.25}{s^2 + 2.25s + 1}$$

poles are at -1.64 and -0.61 ; stable because all poles are in the LHP.

(c) The initial condition is $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and thus

$$\begin{aligned} L^{-1}\{\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0\} &= L^{-1}\left\{\frac{-1}{s^2 + 2.25s + 1}\right\} = L^{-1}\left\{\frac{0.97}{s + 1.64} - \frac{0.97}{s + 0.61}\right\} \\ &= 0.97(e^{-1.64t} - e^{-0.61t}) \end{aligned}$$

and the response due to the external source

$$\begin{aligned} L^{-1}\{\mathbf{H}(s)U(s)\} &= L^{-1}\left\{\frac{s + 0.25}{s^2 + 2.25s + 1} \cdot \frac{5}{s}\right\} = L^{-1}\left\{\frac{1.25}{s} - \frac{4.114}{s + 1.64} + \frac{2.865}{s + 0.61}\right\} \\ &= 1.25 - 4.114e^{-1.64t} + 2.865e^{-0.61t} \end{aligned}$$

The complete response is then given by

$$y(t) = 1.25 - 3.144e^{-1.64t} + 1.895e^{-0.61t}$$

(d)

