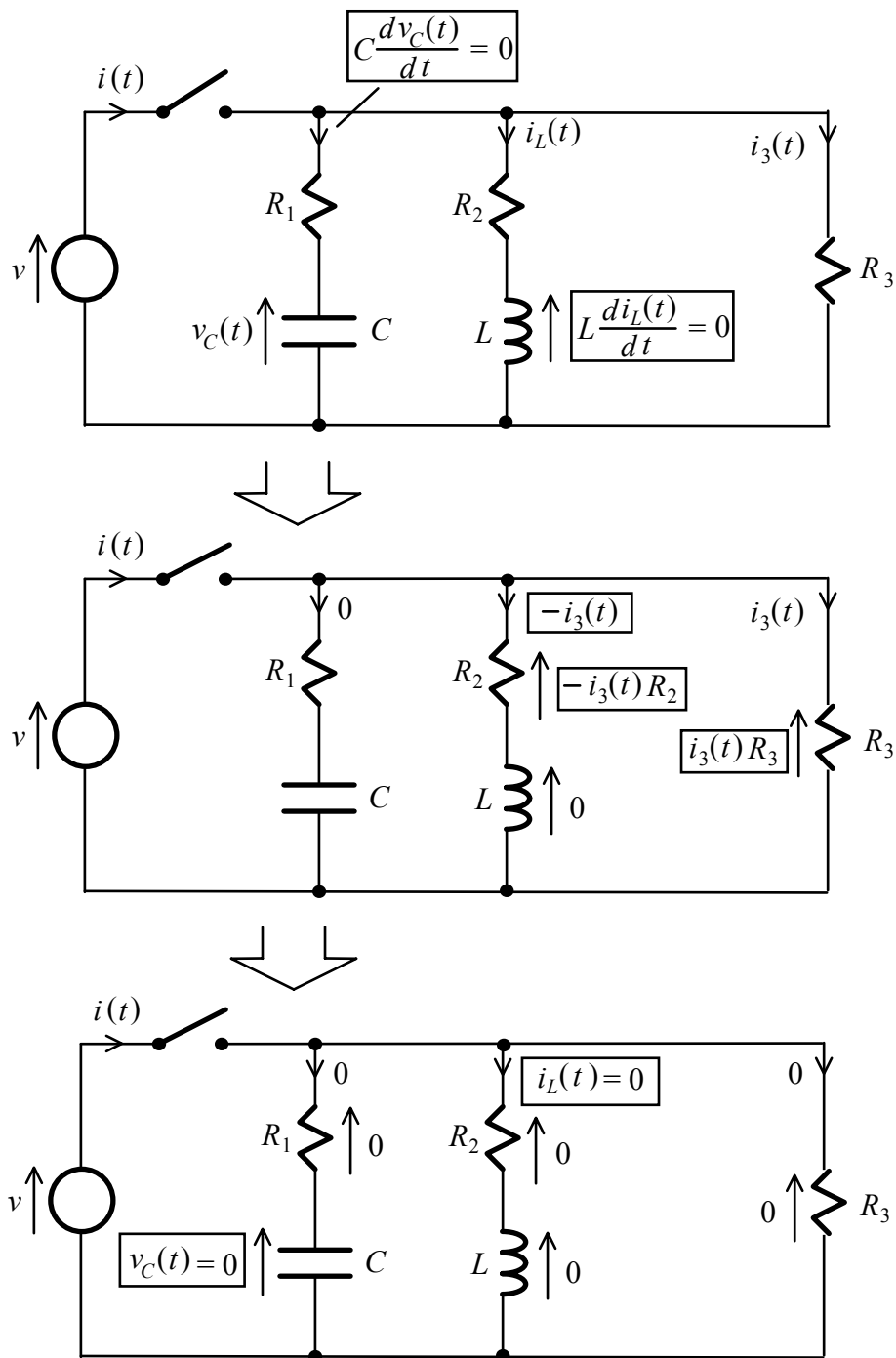


Solution to Tutorial Set 3

Q.1

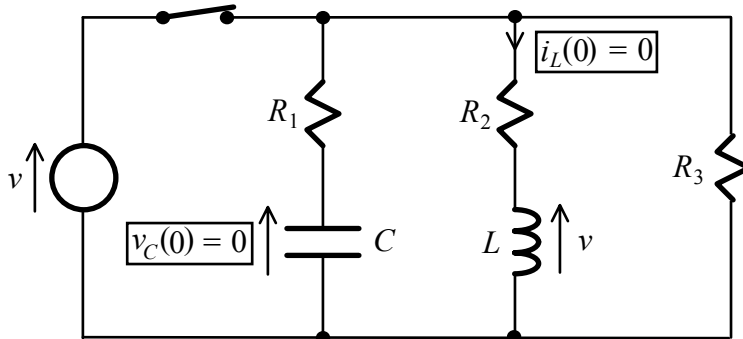
Voltages and currents for $t < 0$

Taking the switches to be in the positions shown starting from $t = -\infty$, all the voltages and currents will have settled down to constant values for practically all $t < 0$.

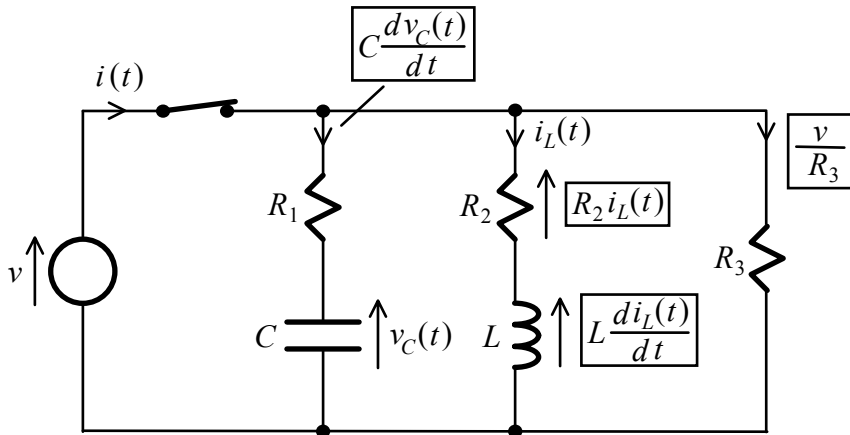


Voltages and currents just after closing switch at $t = 0$

Since the voltage across a capacitor and the current through an inductor must be continuous, the initial conditions are



Voltages and currents for $t \geq 0$



$$v = v_C(t) + CR_1 \frac{dv_C(t)}{dt} \Rightarrow \frac{dv_C(t)}{dt} + \frac{v_C(t)}{CR_1} = \frac{v}{CR_1}$$

$$v = R_2 i_L(t) + L \frac{di_L(t)}{dt} \Rightarrow \frac{di_L(t)}{dt} + \frac{R_2}{L} i_L(t) = \frac{v}{L}$$

The general solutions are

$$v_C(t) = v_{ss}(t) + v_{tr}(t)$$

$$i_L(t) = i_{ss}(t) + i_{tr}(t)$$

The steady state responses are

$$v_{ss}(t) = k_v, t \geq 0$$

$$\frac{dv_{ss}(t)}{dt} + \frac{v_{ss}(t)}{CR_1} = \frac{k_v}{CR_1} = \frac{v}{CR_1} \Rightarrow k_v = v \Rightarrow v_{ss}(t) = v, t \geq 0$$

$$i_{ss}(t) = k_i, t \geq 0$$

$$\frac{di_{ss}(t)}{dt} + \frac{R_2}{L}i_{ss}(t) = \frac{R_2k_i}{L} = \frac{v}{L} \Rightarrow k_i = \frac{v}{R_2} \Rightarrow i_{ss}(t) = \frac{v}{R_2}, t \geq 0$$

The transient responses are given by

$$\frac{dv_{tr}(t)}{dt} + \frac{v_{tr}(t)}{CR_1} = 0 \Rightarrow v_{tr}(t) = h_v e^{-\frac{t}{CR_1}}, t \geq 0$$

$$\frac{di_{tr}(t)}{dt} + \frac{R_2}{L}i_{tr}(t) = 0 \Rightarrow i_{tr}(t) = h_i e^{-\frac{R_2 t}{L}}, t \geq 0$$

Combining and applying initial conditions:

$$v_C(t) = v_{ss}(t) + v_{tr}(t) = v + h_v e^{-\frac{t}{CR_1}}, t \geq 0$$

$$v_C(0) = v + h_v = 0 \Rightarrow h_v = -v \Rightarrow v_C(t) = v - v e^{-\frac{t}{CR_1}}, t \geq 0$$

$$i_L(t) = i_{ss}(t) + i_{tr}(t) = \frac{v}{R_2} + h_i e^{-\frac{R_2 t}{L}}, t \geq 0$$

$$i_L(0) = \frac{v}{R_2} + h_i = 0 \Rightarrow h_i = -\frac{v}{R_2} \Rightarrow i_L(t) = \frac{v}{R_2} - \frac{v}{R_2} e^{-\frac{R_2 t}{L}}, t \geq 0$$

Source current: for $t \geq 0$

$$i(t) = C \frac{dv_C(t)}{dt} + i_L(t) + \frac{v}{R_3} = \frac{Cv}{CR_1} e^{-\frac{t}{CR_1}} + \frac{v}{R_2} - \frac{v}{R_2} e^{-\frac{R_2 t}{L}} + \frac{v}{R_3} = \frac{v}{R_2} + \frac{v}{R_3} + v \left(\frac{e^{-\frac{t}{CR_1}}}{R_1} - \frac{e^{-\frac{R_2 t}{L}}}{R_2} \right)$$

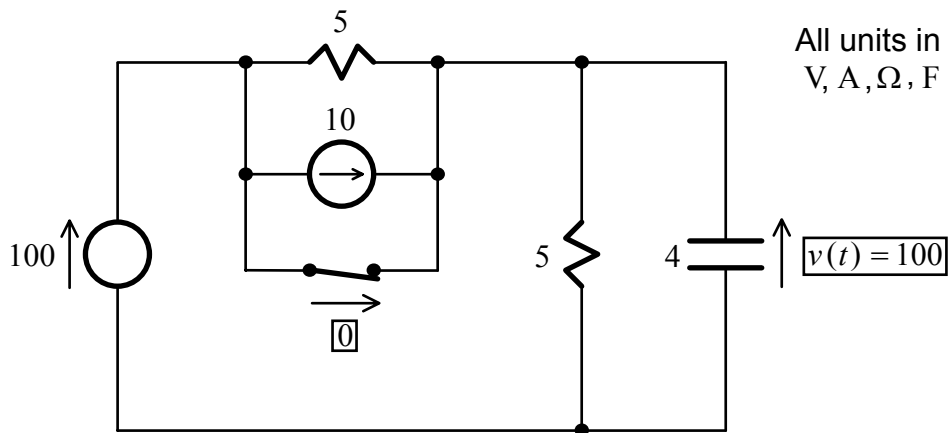
For this to be time independent:

$$R_1 = R_2$$

$$CR_1 = \frac{L}{R_2} \Rightarrow R_1^2 = R_2^2 = \frac{L}{C}$$

Q.2

Voltages and currents for $t < 0$ before the switch is opened

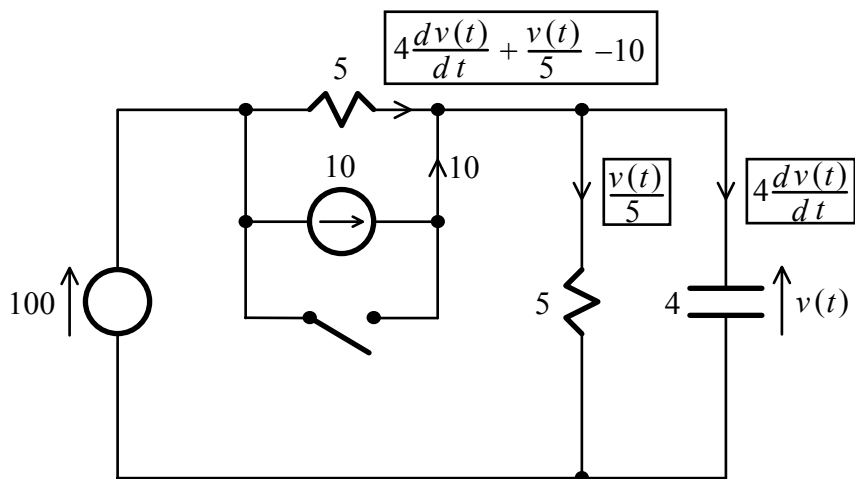


$v(t)$ at $t = 0$ just after the switch is opened

Since the voltage across the capacitor must be continuous, $v(0)$ must be

$$v(0) = 100$$

$v(t)$ for $t \geq 0$ after the switch is opened



$$100 = 5 \left[4 \frac{dv(t)}{dt} + \frac{v(t)}{5} - 10 \right] + v(t) \Rightarrow 20 \frac{dv(t)}{dt} + 2v(t) = 150$$

The general solution of this is given by

$$v(t) = v_{ss}(t) + v_{tr}(t)$$

$v_{ss}(t)$ is the steady state response and can be found by trying

$$v_{ss}(t) = k, t \geq 0$$

$$20 \frac{dv_{ss}(t)}{dt} + 2v_{ss}(t) = 0 + 2k = 150 \Rightarrow k = 75 \Rightarrow v_{ss}(t) = 75, t \geq 0$$

$v_{tr}(t)$ is the transient response and is equal to the general solution of

$$20 \frac{dv_{tr}(t)}{dt} + 2v_{tr}(t) = 0 \Rightarrow v_{tr}(t) = he^{-\frac{t}{10}}, t \geq 0$$

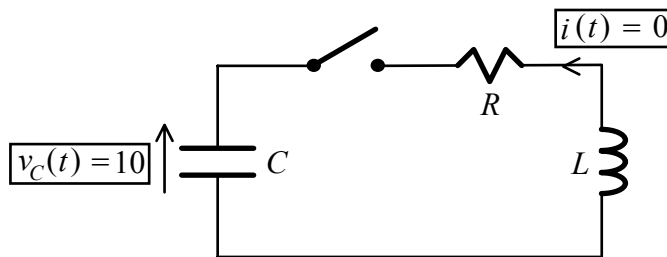
Combining:

$$v(t) = v_{ss}(t) + v_{tr}(t) = 75 + he^{-\frac{t}{10}}, t \geq 0$$

$$v(0) = 75 + h = 100 \Rightarrow h = 25 \Rightarrow v(t) = 75 + 25e^{-\frac{t}{10}}, t \geq 0$$

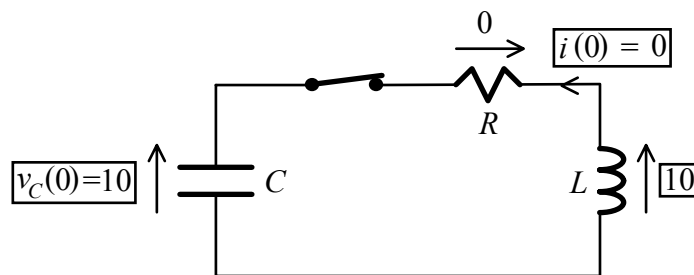
Q.3

(a) Voltage and current before closing switch

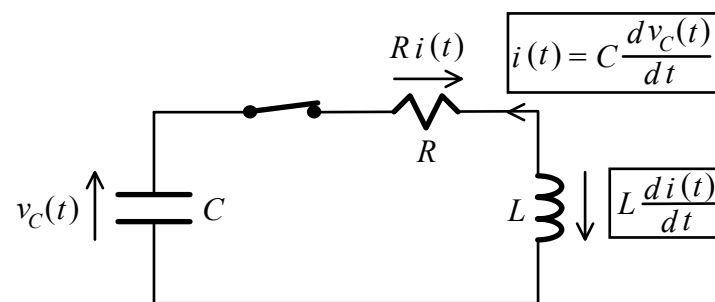


(b) Voltages and current just after closing switch at $t = 0$

Since the voltage across a capacitor and the current through an inductor must be continuous, the initial conditions are



(c) Governing differential equation for current for $t \geq 0$



$$L \frac{di(t)}{dt} + Ri(t) + v_C(t) = 0 \Rightarrow \frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{L} \frac{dv_C(t)}{dt} = 0$$

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0, t \geq 0$$

with initial conditions

$$i(0) = 0$$

$$v_C(0) = 10 \Rightarrow L \left. \frac{di(t)}{dt} \right|_{t=0} = -10 \Rightarrow \left. \frac{di(t)}{dt} \right|_{t=0} = -\frac{10}{L}$$

Substituting

$$\text{Resonant frequency} = \omega = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega L}{R} = \sqrt{\frac{L}{R^2 C}}$$

the governing differential equation is

$$\frac{d^2i(t)}{dt^2} + \frac{\omega}{Q} \frac{di(t)}{dt} + \omega^2 i(t) = 0, t \geq 0$$

(d) *Overdamped situation when $Q < \frac{1}{2}$*

The roots of the polynomial

$$\left. \frac{d^2i(t)}{dt^2} + \frac{\omega}{Q} \frac{di(t)}{dt} + \omega^2 i(t) \right|_{\substack{di(t) \\ dt} \text{ replaced by } z} = z^2 + \frac{\omega}{Q} z + \omega^2$$

are

$$z_1, z_2 = \frac{-\frac{\omega}{Q} \pm \sqrt{\frac{\omega^2}{Q^2} - 4\omega^2}}{2} = \frac{-\omega}{2Q} \left(1 \pm \sqrt{1 - 4Q^2} \right)$$

When $Q < \frac{1}{2}$, both roots are real, negative and distinct. Thus:

$$i(t) = k_1 e^{z_1 t} + k_2 e^{z_2 t}, t \geq 0$$

$$z_1 = \frac{-\omega}{2Q} \left(1 - \sqrt{1 - 4Q^2} \right) < 0$$

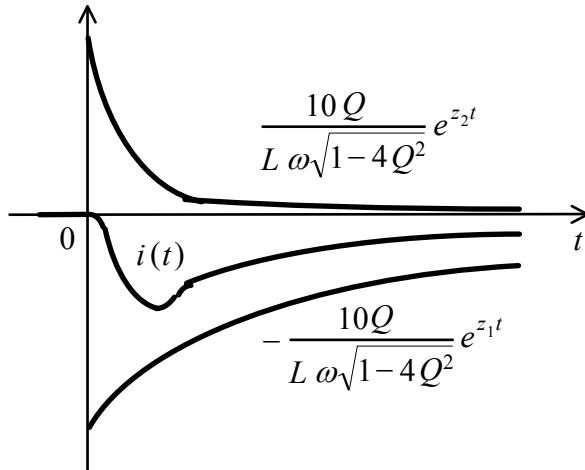
$$z_2 = \frac{-\omega}{2Q} \left(1 + \sqrt{1 - 4Q^2} \right) < z_1$$

Using initial conditions:

$$i(0) = 0 \Rightarrow k_1 + k_2 = 0 \Rightarrow k_1 = -k_2$$

$$\left. \frac{di(t)}{dt} \right|_{t=0} = -\frac{10}{L} \Rightarrow k_1 z_1 + k_2 z_2 = -k_2(z_1 - z_2) = -\frac{k_2 \omega}{Q} \sqrt{1-4Q^2} = -\frac{10}{L}$$

$$i(t) = k_2(e^{z_2 t} - e^{z_1 t}) = \frac{10Q}{L\omega\sqrt{1-4Q^2}}(e^{z_2 t} - e^{z_1 t}), t \geq 0$$



(e) Underdamped situation when $Q > \frac{1}{2}$

When $Q > \frac{1}{2}$, the two roots form a complex pair:

$$z_1 = \frac{-\omega}{2Q} \left(1 - j\sqrt{4Q^2 - 1}\right) = -\frac{\omega}{2Q} + j\omega\sqrt{1 - \frac{1}{4Q^2}}$$

$$z_2 = \frac{-\omega}{2Q} \left(1 + j\sqrt{4Q^2 - 1}\right) = -\frac{\omega}{2Q} - j\omega\sqrt{1 - \frac{1}{4Q^2}} = z_1^*$$

$$i(t) = h_1 e^{z_1 t} + h_2 e^{z_2 t}, t \geq 0$$

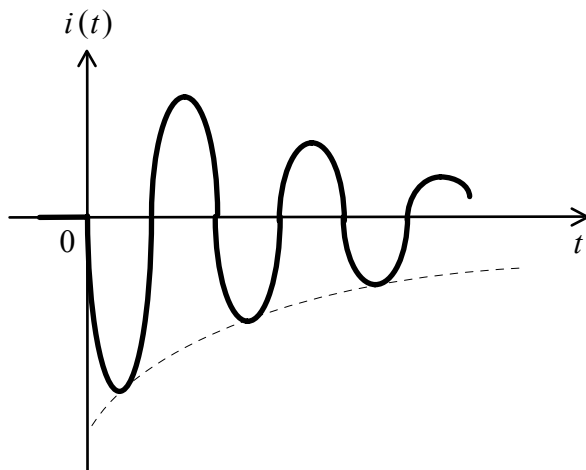
Using initial conditions:

$$i(0) = 0 \Rightarrow h_1 + h_2 = 0 \Rightarrow h_1 = -h_2$$

$$\left. \frac{di(t)}{dt} \right|_{t=0} = -\frac{10}{L} \Rightarrow h_1 z_1 + h_2 z_2 = -h_2(z_1 - z_2) = -jh_2 2\omega\sqrt{1 - \frac{1}{4Q^2}} = -\frac{10}{L}$$

$$i(t) = h_2(e^{z_2 t} - e^{z_1 t}) = \frac{-j10}{2L\omega\sqrt{1 - \frac{1}{4Q^2}}} \left[e^{\left(-\frac{\omega}{2Q} - j\omega\sqrt{1 - \frac{1}{4Q^2}}\right)t} - e^{\left(-\frac{\omega}{2Q} + j\omega\sqrt{1 - \frac{1}{4Q^2}}\right)t} \right]$$

$$= \frac{-j10e^{-\frac{\omega}{2Q}t}}{2L\omega\sqrt{1 - \frac{1}{4Q^2}}} \left[e^{-j\omega\sqrt{1 - \frac{1}{4Q^2}}t} - e^{j\omega\sqrt{1 - \frac{1}{4Q^2}}t} \right] = \frac{-10e^{-\frac{\omega}{2Q}t}}{L\omega\sqrt{1 - \frac{1}{4Q^2}}} \sin\left(\omega\sqrt{1 - \frac{1}{4Q^2}}t\right), t \geq 0$$

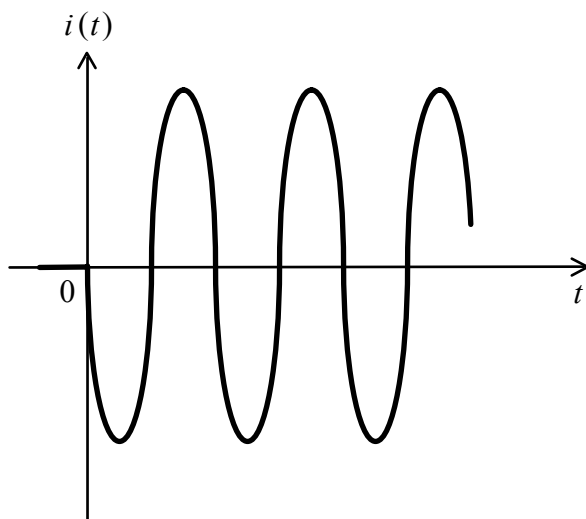


(e) Undamped situation when $R = 0$

When $R = 0$, the Q factor and current is

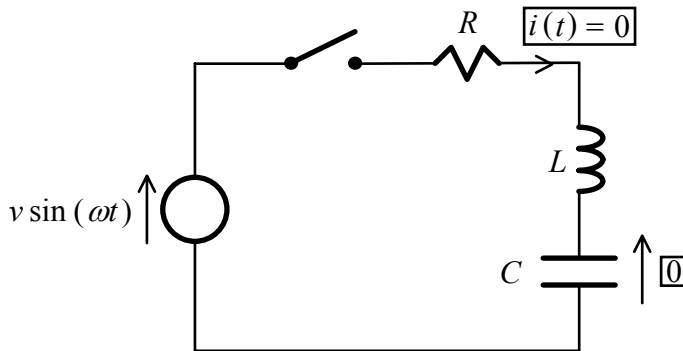
$$Q = \frac{\omega L}{R} = \infty$$

$$i(t) = \frac{-10e^{-\frac{\omega}{2Q}t}}{L\omega\sqrt{1-\frac{1}{4Q^2}}} \sin\left(\omega\sqrt{1-\frac{1}{4Q^2}}t\right) = \frac{-10}{L\omega} \sin(\omega t), t \geq 0$$



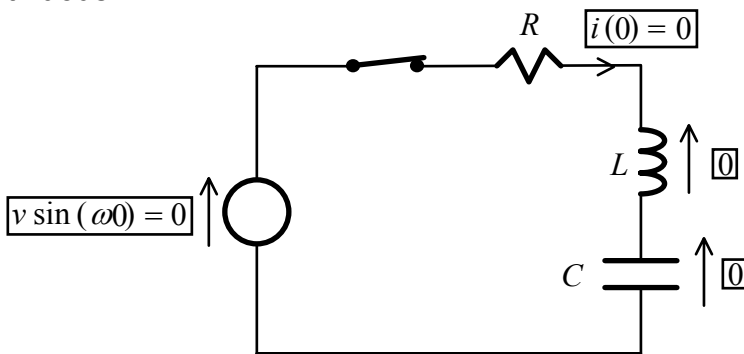
Q.4

Voltages and current for $t < 0$ before the switch is closed

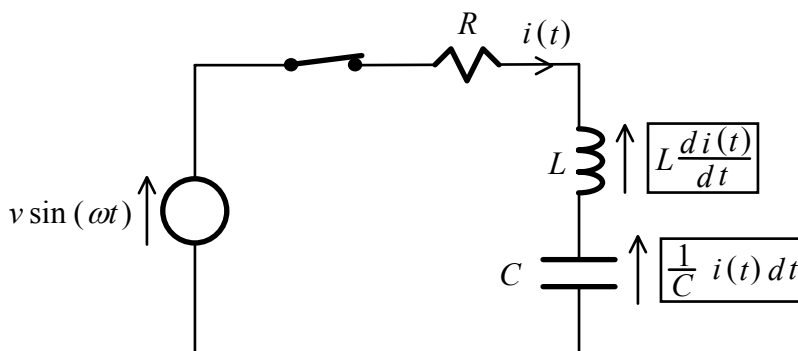


Voltages and current at $t = 0$ just after the switch is closed

As the voltage across the capacitor and the current through the inductor must be continuous:



Current for $t \geq 0$ after the switch is closed



$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int i(t) dt = v \sin(\omega t) = \text{Re}[-jve^{j\omega t}]$$

$$L \frac{d^2i(t)}{dt^2} + R \frac{di(t)}{dt} + \frac{1}{C} i(t) = \text{Re}[\omega ve^{j\omega t}]$$

with initial conditions

$$i(0) = 0$$

$$\text{Initial voltage across inductor} = 0 \Rightarrow L \left. \frac{di(t)}{dt} \right|_{t=0} = 0$$

Steady state current

The steady state response $i_{ss}(t)$ is given by

$$i_{ss}(t) = \text{Re}[I_{ss}e^{j\omega t}], t \geq 0$$

$$L \frac{d^2 \text{Re}[I_{ss}e^{j\omega t}]}{dt^2} + R \frac{d \text{Re}[I_{ss}e^{j\omega t}]}{dt} + \frac{1}{C} \text{Re}[I_{ss}e^{j\omega t}] = \text{Re}[\omega v e^{j\omega t}]$$

$$\text{Re} \left[\left(L \frac{d^2 e^{j\omega t}}{dt^2} + R \frac{d e^{j\omega t}}{dt} + \frac{e^{j\omega t}}{C} \right) I_{ss} \right] = \text{Re} \left[\left(j^2 \omega^2 L + j\omega R + \frac{1}{C} \right) I_{ss} e^{j\omega t} \right] = \text{Re}[\omega v e^{j\omega t}]$$

$$\left(j^2 \omega^2 L + j\omega R + \frac{1}{C} \right) I_{ss} = \omega v \Rightarrow I_{ss} = \frac{v}{j \left(j\omega L + R + \frac{1}{jC} \right)} = \frac{v}{jZ}$$

$$i_{ss}(t) = \text{Re} \left[\frac{v}{jZ} e^{j\omega t} \right] = \text{Re} \left[\frac{v}{j|Z|} e^{j\text{Arg}(Z)} e^{j\omega t} \right] = \frac{v}{|Z|} \sin[\omega t - \text{Arg}(Z)]$$

$$= \frac{v}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}} \sin \left[\omega t - \tan^{-1} \left(\frac{\omega L - \frac{1}{\omega C}}{R} \right) \right], t \geq 0$$