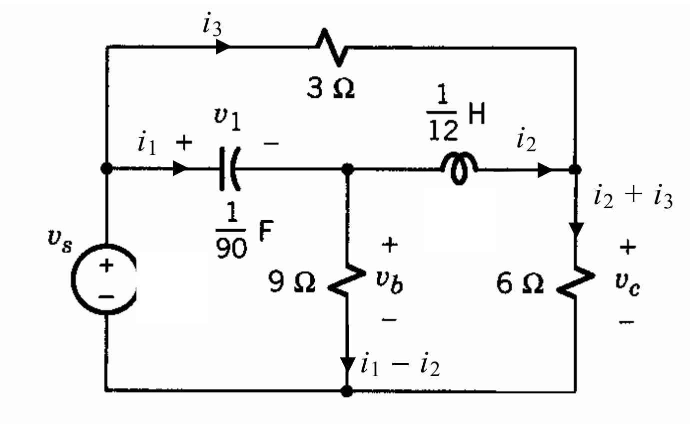
Solution to Tutorial Set 4

Q.1 (a)



The voltage over the inductor is given by

$$v_2 = L\frac{di_2}{dt} = \frac{1}{12}\frac{di_2}{dt}$$

The current through the capacitor is

$$i_1 = C\frac{dv_1}{dt} = \frac{1}{90}\frac{dv_1}{dt}$$

The current through 3Ω is

$$i_3 = \frac{v_1 + v_2}{3} = \frac{v_1 + \frac{1}{12}\frac{di_2}{dt}}{3} = \frac{1}{3}v_1 + \frac{1}{36}\frac{di_2}{dt}$$

Applying KVL to the outer loop, we have

$$v_s = 3i_3 + 6(i_2 + i_3) = 6i_2 + 9i_3 = 6i_2 + 9\left(\frac{1}{3}v_1 + \frac{1}{36}\frac{di_2}{dt}\right) = 3v_1 + 6i_2 + \frac{1}{4}\frac{di_2}{dt}$$

$$\Rightarrow \quad \frac{di_2}{dt} = -12v_1 - 24i_2 + 4v_s$$

Applying KVL to the left-lower loop, we have

$$v_s = v_1 + 9(i_1 - i_2) = v_1 + 9\left(\frac{1}{90}\frac{dv_1}{dt} - i_2\right) = v_1 + \frac{1}{10}\frac{dv_1}{dt} - 9i_2$$

$$\Rightarrow \quad \frac{dv_1}{dt} = -10v_1 + 90i_2 + 10v_s$$

$$\Rightarrow \quad v_b = -v_1 + v_s$$

$$\Rightarrow \quad v_c = 6(i_2 + i_3) = 6i_2 + 6\left(\frac{1}{3}v_1 + \frac{1}{36}\frac{di_2}{dt}\right) = 6i_2 + 2v_1 + \frac{1}{6}\frac{di_2}{dt}$$

$$= 6i_2 + 2v_1 + \frac{1}{6}\left(-12v_1 - 24i_2 + 4v_s\right) = 2i_2 + \frac{2}{3}v_s$$

Putting everything together, we have

$$\begin{pmatrix} v_1' \\ i_2' \end{pmatrix} = \begin{bmatrix} -10 & 90 \\ -12 & -24 \end{bmatrix}\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} + \begin{bmatrix} 10 \\ 4 \end{bmatrix} v_s, \qquad \begin{pmatrix} v_b \\ v_c \end{pmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} + \begin{bmatrix} 1 \\ 2/3 \end{bmatrix} v_s$$

(b) The complete response is given by

$$\mathbf{Y(s)} = \left[\mathbf{C(sI-A)^{-1}B} + \mathbf{D}\right]\mathbf{U(s)} + \mathbf{C(sI-A)^{-1}X_0}$$

$$= \left\{\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} s+10 & -90 \\ 12 & s+24 \end{bmatrix}^{-1}\begin{bmatrix} 10 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}\right\}\cdot\frac{1}{s} + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} s+10 & -90 \\ 12 & s+24 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{pmatrix} \dfrac{s^2+24s+720}{s(s^2+34s+1320)} \\ \dfrac{2s^2+92s+2160}{3s(s^2+34s+1320)} \end{pmatrix} + \begin{pmatrix} \dfrac{-s-204}{s^2+34s+1320} \\ \dfrac{4s+16}{s^2+34s+1320} \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} v_b \\ v_c \end{pmatrix} = L^{-1}\left\{\begin{pmatrix} \dfrac{s^2+24s+720}{s(s^2+34s+1320)} \\ \dfrac{2s^2+92s+2160}{3s(s^2+34s+1320)} \end{pmatrix}\right\} + L^{-1}\left\{\begin{pmatrix} \dfrac{-s-204}{s^2+34s+1320} \\ \dfrac{4s+16}{s^2+34s+1320} \end{pmatrix}\right\}$$

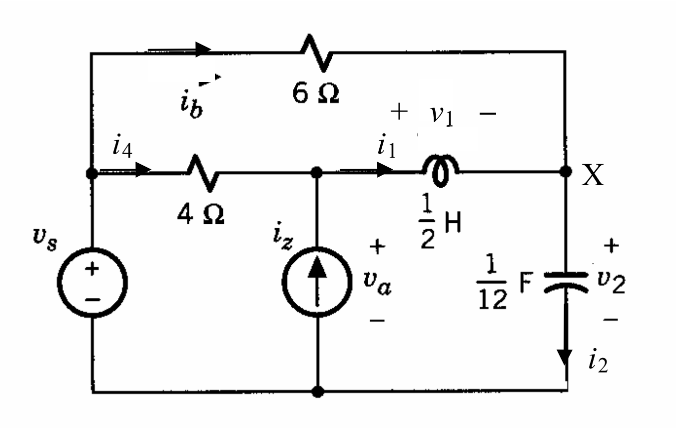
(c) The transfer function of the circuit is

$$\mathbf{H(s)} = \mathbf{C(sI-A)^{-1}B} + \mathbf{D} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} s+10 & -90 \\ 12 & s+24 \end{bmatrix}^{-1}\begin{bmatrix} 10 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2/3 \end{bmatrix}$$

$$= \begin{pmatrix} \dfrac{s^2+24s+720}{s^2+34s+1320} \\ \dfrac{2s^2+92s+2160}{3(s^2+34s+1320)} \end{pmatrix}$$

The poles of the system are roots of $s^2+34s+1320$. They are at $-17 \pm j32.11$, respectively. The system is stable as the poles are in the open left-half plane.

Q.2 (a)



Obviously,

$$v_1 = \frac{1}{2}\frac{di_1}{dt}, \qquad i_2 = \frac{1}{12}\frac{dv_2}{dt}, \qquad i_b = \frac{v_s - v_2}{6}, \qquad i_4 = i_1 - i_z$$

Applying KCL to node X, we have

$$\frac{1}{12}\frac{dv_2}{dt} = i_2 = i_1 + i_b = i_1 + \frac{v_s - v_2}{6} \qquad \Rightarrow \qquad \frac{dv_2}{dt} = 12i_1 - 2v_2 + 2v_s$$

Applying KVL to the loop consisting of the voltage source, the 4Ω resistor, the inductor and the capacitor, we obtain

$$v_s = 4i_4 + v_1 + v_2 = 4(i_1 - i_z) + \frac{1}{2}\frac{di_1}{dt} + v_2 \quad \Rightarrow \quad \frac{di_1}{dt} = -8i_1 - 2v_2 + 2v_s + 8i_z$$

$$\Rightarrow \quad v_a = v_s - 4i_4 = v_s - 4(i_1 - i_z) = -4i_1 + v_s + 4i_z$$

Putting everything together, we have

$$\begin{pmatrix} i_1' \\ v_2' \end{pmatrix} = \begin{bmatrix} -8 & -2 \\ 12 & -2 \end{bmatrix}\begin{pmatrix} i_1 \\ v_2 \end{pmatrix} + \begin{bmatrix} 2 & 8 \\ 2 & 0 \end{bmatrix}\begin{pmatrix} v_s \\ i_z \end{pmatrix}, \qquad \begin{pmatrix} v_a \\ i_b \end{pmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -1/6 \end{bmatrix}\begin{pmatrix} i_1 \\ v_2 \end{pmatrix} + + \begin{bmatrix} 1 & 4 \\ 1/6 & 0 \end{bmatrix}\begin{pmatrix} v_s \\ i_z \end{pmatrix}$$

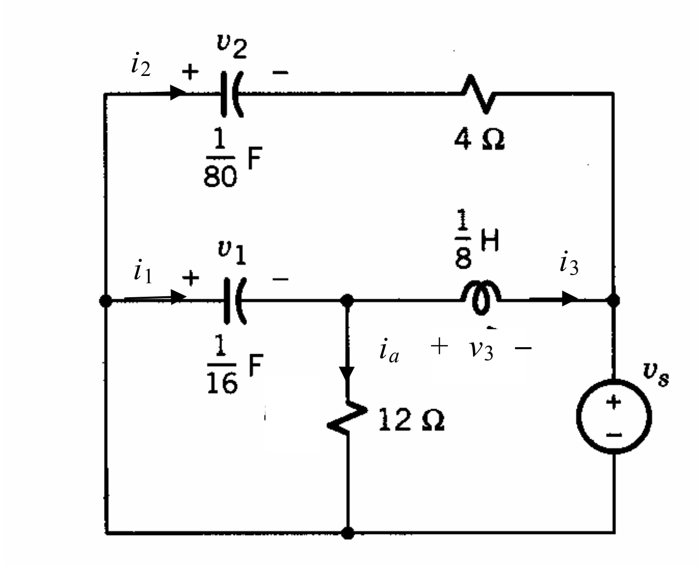
(b) The transfer function of the circuit,

$$\mathbf{H(s)} = \mathbf{C(sI-A)^{-1}B+D} = \begin{bmatrix} -4 & 0 \\ 0 & -1/6 \end{bmatrix} \begin{bmatrix} s+8 & 2 \\ -12 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 8 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 1/6 & 0 \end{bmatrix}$$

$$= \frac{1}{s^2+10s+40} \begin{pmatrix} s^2+2s+40 & 4s^2+8s+96 \\ (s^2+8s)/6 & -16 \end{pmatrix}$$

The poles of the system are given by the roots of the polynomial $s^2+10s+40$, which are at $-5 \pm j3.873$, respectively. Hence, the system is stable.

Q.3 (a)



First, there are three energy-storing elements in the circuit. In general, we have three state variables. It is obvious that

$$i_1 = \frac{1}{16}\frac{dv_1}{dt}, \qquad i_2 = \frac{1}{80}\frac{dv_2}{dt}, \qquad v_3 = \frac{1}{8}\frac{di_3}{dt}, \qquad i_a = i_1 - i_3$$

We need to find three equations as there are three state variables. Applying KVL to the outer loop, we have

$$0 = v_2 + 4i_2 + v_s = v_2 + \frac{1}{20}\frac{dv_2}{dt} + v_s \qquad \Rightarrow \qquad \frac{dv_2}{dt} = -20v_2 - 20v_s$$

Applying KVL to the left-lower loop, we obtain

$$0 = v_1 + 12i_a = v_1 + 12(i_1 - i_3) = v_1 - 12i_3 + 12\cdot\frac{1}{16}\frac{dv_1}{dt} \qquad \Rightarrow \qquad \frac{dv_1}{dt} = -\frac{4}{3}v_1 + 16i_3$$

Once again, applying KVL to the lower half loop, we have

$$0 = v_1 + v_3 + v_s = v_1 + \frac{1}{8}\frac{di_3}{dt} + v_s \qquad \Rightarrow \qquad \frac{di_3}{dt} = -8v_1 - 8v_s$$

The output equation can be obtained as

$$i_a = -\frac{1}{12}v_1$$

Putting everything in a compact matrix form,

$$\begin{pmatrix} v_1' \\ v_2' \\ i_3' \end{pmatrix} = \begin{bmatrix} -4/3 & 0 & 16 \\ 0 & -20 & 0 \\ -8 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ i_3 \end{pmatrix} + \begin{bmatrix} 0 \\ -20 \\ -8 \end{bmatrix} v_s, \qquad i_a = \begin{bmatrix} -1/12 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ i_3 \end{pmatrix} + 0 \cdot v_s$$

(b) Entering the matrices in MATLAB and using the m-function `ss2tf`, i.e.,

```
>> format short e     % To display data in terms of powers of 10
>> [NUM,DEN] = SS2TF(A,B,C,D,1)
```

we obtain the numerator and denominator of the transfer function,

```
NUM =

                0   3.5527e-015   1.0667e+001   2.1333e+002

DEN =

      1.0000e+000   2.1333e+001   1.5467e+002   2.5600e+003
```

Thus, the transfer function is given by

$$\mathbf{H(s)} = \frac{10.667s + 21.333}{s^3 + 21.333s^2 + 154.67s + 2560}$$

The poles of the system can be computed as

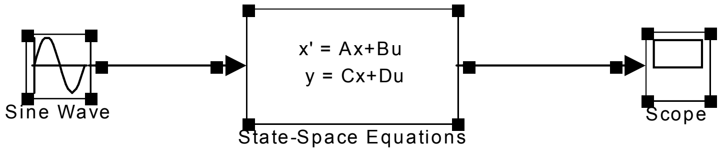
```
>> format        % back to default to display in a normal way.
>> roots(DEN)
ans =

     -20.0000
     -0.6667 +11.2940i
     -0.6667 -11.2940i
```

They are in the open left-half complex plane. Hence, the system is stable. The zeros of the system can be computed similarly,

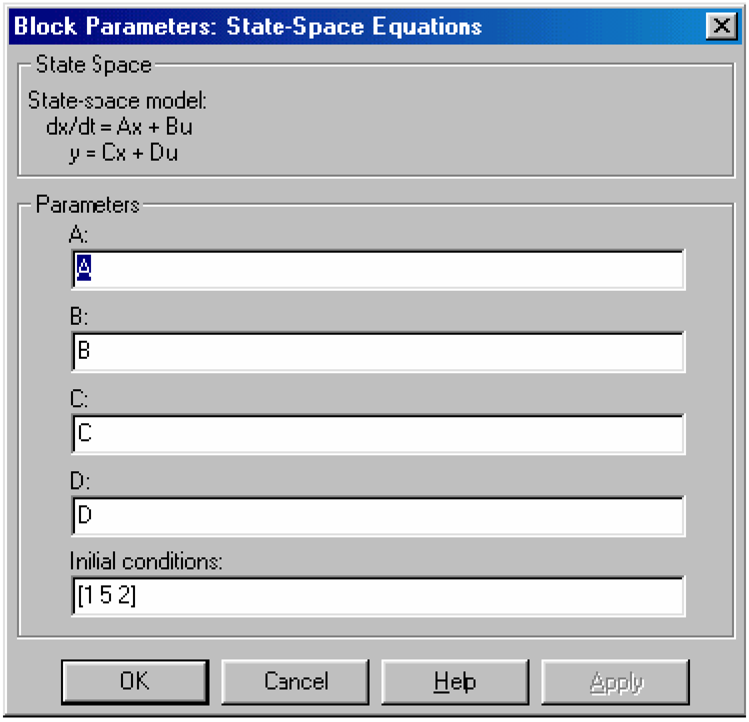
```
>> roots(NUM)
ans =

     -20.0000
```

The system has a zero as `-20`.

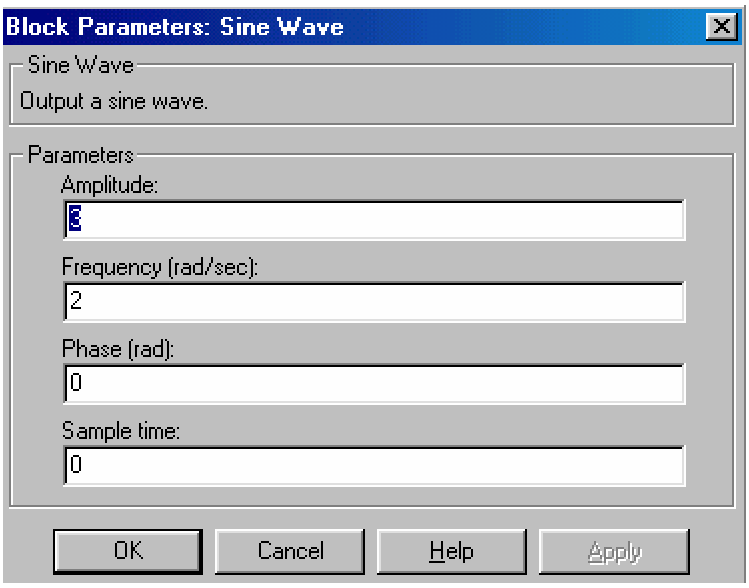
(c) Build up a SIMULINK model,



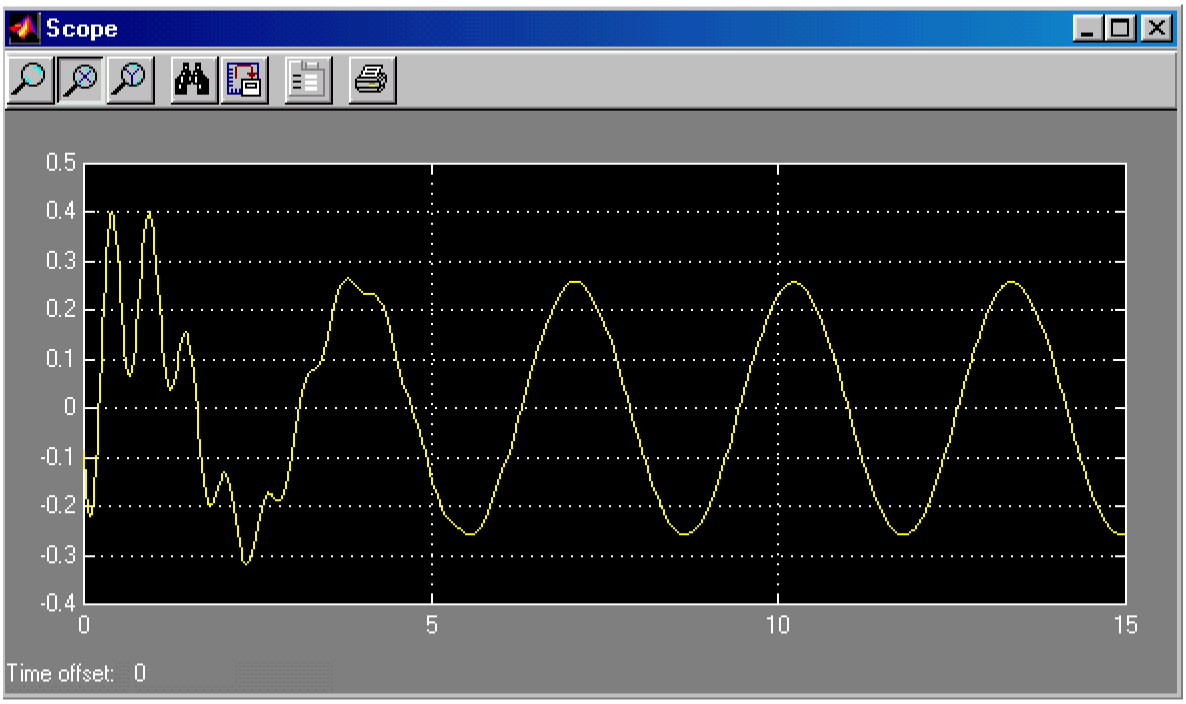
The state space block is entered as follows,



The source block is entered as,



The complete response of the circuit can be viewed from the 'Scope'. It is given as follows,



Transient response can be observed in the first few seconds. Eventually, the circuit settles down to the steady state response, which is a sinusoidal wave.