EE2462 Engineering Math III (Part 1)

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Course Outline

Series and Power Series: Sequences and series; convergence and divergence; a test for divergence; comparison tests for positive series; the ratio test for positive series; absolute convergence; power series.

Special Functions: Bessel′s equation and Bessel functions; the Gamma function; solution of Bessel′s equation in terms of Gamma function; Modified Bessel′s equations; Applications of Bessel′s functions; Legendr′s equation; Legendre polynomials and their properties.

Partial Differential Equations: Boundary value problem in partial differential equations; wave equation; heat equation; Laplace equation; Poisson′s equation; Dirichlet and Nuemann Problems. Solutions to wave and heat equations using method of separation of variables.

Lectures

Lectures will follow closely (but not 100%) the materials in the lecture notes (available at http://vlab.ee.nus.edu.sg/~bmchen).

However, certain parts of the lecture notes will not be covered and examined and this will be made known during the classes.

Attendance is essential.

ASK any question at any time during the lecture.

Tutorials

The tutorials will start on Week 4 of the semester (again, tutorial sets can be downloaded from http://vlab.ee.nus.edu.sg/~bmchen).

Solutions to Tutorial Sets 1, 3 and 4 will be available from my web site one week after they are conducted. Tutorial Set 2 is an interactive one.

Although you should make an effort to attempt each question before the tutorial, it is not necessary to finish all the questions.

Some of the questions are straightforward, but quite a few are difficult and meant to serve as a platform for the introduction of new concepts.

ASK your tutor any question related to the tutorials and the course.

Reference Textbooks

- **P. V. O′Neil, *Advanced Engineering Mathematics*, Any Ed., PWS.**

- **E. Kreyszig, *Advanced Engineering Mathematics*, Any Ed., Wiley.**

Sequences and Series

A sequence consists of a set numbers that is arranged in order. For example,

$$\{n\} \ : \ 1, \quad 2, \quad \cdots, \quad n, \quad \cdots$$

$$\left\{\frac{1}{2^n}\right\} \ : \ \frac{1}{2}, \ \frac{1}{4}, \ \cdots, \ \frac{1}{2^n}, \ \cdots$$

$$\left\{\frac{1}{n}\right\} \ : \ 1, \ \frac{1}{2}, \ \cdots, \ \frac{1}{n}, \ \cdots$$

In general, a sequence has the following form

$$\{u_n\} \ : \ u_1, \quad u_2, \quad \cdots, \quad u_n, \quad \cdots$$

For any given sequence, we define an array (or called a series)

$$\sum u_n = u_1 + u_2 + \cdots + u_n + \cdots$$

Let us define

$$s_n = \sum_{i=1}^{n} u_i = u_1 + u_2 + \cdots + u_n$$

This partial sum forms a new sequence { s_n }. If, as n increases and tends to infinity the sequence of numbers s_n approaches a finite limit $\boldsymbol{L}$, we say that the series

$$\sum u_i = u_1 + u_2 + \cdots + u_n + \cdots$$

converges. And we write

$$\lim_{n \to \infty} s_n = \sum_{i=1}^{\infty} u_i = L.$$

We say that the infinite series converges to L and that L is the value of the series. If the sequence does not approach a limit, the series is **divergent** and we do not assign any value to it.

A Divergence Test

$$s_n = \sum_{i=1}^{n} u_i = u_1 + u_2 + \cdots + u_{n-1} + u_n = s_{n-1} + u_n$$

Consider a series

$$\sum u_i = u_1 + u_2 + \cdots + u_n + \cdots$$

If it converges, then we have

$$\lim_{n\to\infty} s_n = L \quad \text{and} \quad \lim_{n\to\infty} s_{n-1} = L.$$

and hence

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0$$

Thus, if u_n does not tend to zero as n becomes infinite, the series

$$\sum u_i = u_1 + u_2 + \cdots + u_n + \cdots$$

is divergent.

Example 1: Show that the series with

$$u_n = \frac{n}{2n+1}, \qquad \text{i.e., the series } \sum u_n$$

diverges.

Solution: Check that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$$

Hence, it diverges.

Example 2. Show that the series with odd terms equal to $(n+1)/n$ and even terms equal to $1/n$ diverges.

Solution: For this case, the odd terms $(n+1)/n$ are actually approaching to a nonzero value 1. The limit of u_n cannot be zero and hence it diverges.

Comparison Test for Positive Series

A series $\Sigma\, u_n$ is said to be positive if $u_n \geq 0$. The following results are called comparison test for positive series:

1. Let $\Sigma\, v_n$ be a positive series, which converges. If $0 \leq u_n \leq v_n$ for all n, then the series $\Sigma\, u_n$ converges.

2. Let $\Sigma\, V_n$ be a positive series, which diverges. If $u_n \geq V_n$ for all n, then the series $\Sigma\, u_n$ diverges.

Ratio Test for Positive Series

For positive series $\Sigma\, u_n$ with

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = T$$

then

1. The series converges if $T < 1$.
2. The series diverges if $T > 1$.
3. No conclusion can be made if $T = 1$. Further test is needed.

Example: Test the series with $u_n = \dfrac{(n-1)!}{n^{n-1}}$ for convergence.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n!}{(n+1)^n} \cdot \frac{n^{n-1}}{(n-1)!} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$$

Hence, it converges.

Absolute Convergence

Consider a general series $\Sigma\, u_n$

1. If $\Sigma\, |u_n|$ converges, then $\Sigma\, u_n$ converges. Actually, we will say that the series $\Sigma\, u_n$ **converges absolutely** or has an absolute convergence.

2. If $\Sigma\, |u_n|$ diverges, then $\Sigma\, u_n$ may converge or diverge. If $\Sigma\, u_n$ converges, we say that it **converges conditionally**.

Note that $\Sigma\, |u_n|$ is a positive series and its convergence can be test using the ratio test.

Power Series

Any infinite series of the form

$$A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots$$

is called a power series, which can be written as a normal series,

$$\sum_{n=0}^{\infty} u_n(x) \qquad \text{where} \quad u_n(x) = A_n(x-a)^n$$

Theorem (see lecture notes for proof): Consider the above power series. If

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = L \neq 0$$

then the power series converges absolutely for any x such that $|x-a| < \frac{1}{L}$

and the power series diverges for any x such that $|x-a| > \frac{1}{L}$.

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Example: Find the open interval of absolute convergence of the power series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots$$

Solution: Using the theorem for power series, we have

$$A_n = \frac{1}{n} \quad \text{and} \quad a = 0 \quad \Rightarrow \quad \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1 = L$$

Hence, the power series converges absolutely for all

$$|x| < 1 \quad \text{or} \quad -1 < x < 1$$

and diverges for all

$$|x| > 1$$

Bessel′s Equation

The following second order differential equation,

$$x^2 y'' + xy' + \left(x^2 - v^2\right)y = 0$$

is called Bessel′s equation of order v. Note that Bessel′s equation is a 2nd order differential equation. It can be used to model quite a number of problems in engineering such as the model of the **displacement of a suspended chain**, the **critical length of a vertical rod**, and the **skin effect of a circular wire in AC circuits**. The first application will be covered in details in the class later on.

In general, it is very difficult to derive a closed-form solutions to differential equations. As expected, the solution to the above Bessel′s equation cannot be expressed in terms of some "nice" forms.

Solution to Bessel′s Equation (Bessel Function of the First Kind)

The solution to the Bessel′s equation is normally expressed in terms of a power series, which has a special name called **Frobenius series.** Such a method is called **Method of Frobenius**. We define a power series (a Frobenius series),

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

where r and c_n are free parameters. Without loss of any generality, let $c_0 \neq 0$.

Next, we will try to determine these parameters r and c_n such that the above Frobenius series is a solution of the Bessel′s equation, i.e., it will satisfy the Bessel′s equation,

$$x^2 y'' + xy' + \left(x^2 - v^2\right)y = 0$$

Now, assume that the Frobenius series $y(x)$ is indeed a solution to the Bessel's equation. We compute

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

Then, the Bessel's equation gives

$$0 = x^2 y'' + xy' + \left(x^2 - v^2\right) y$$

$$= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r}$$

$$+ \sum_{n=0}^{\infty} c_n (n+r) x^{n+r}$$

$$- \sum_{n=0}^{\infty} c_n v^2 x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2}$$

Let

$$m = n+2 \quad \Rightarrow \quad n = m-2$$

$$n = 0 \quad \Rightarrow \quad m = 2$$

$$n = \infty \quad \Rightarrow \quad m = \infty$$

$$\sum_{n=0}^{\infty} c_n x^{n+r+2} = \sum_{m=2}^{\infty} c_{m-2} x^{m+r}$$

$$= \sum_{n=2}^{\infty} c_{n-2} x^{n+r}$$

Thus,

$$0 = x^2 y'' + xy' + \left(x^2 - v^2\right)y$$

$$= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} c_n (n+r)x^{n+r} - \sum_{n=0}^{\infty} c_n v^2 x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r}$$

$$= \sum_{n=0}^{\infty} c_n (n+r)^2 x^{n+r} - \sum_{n=0}^{\infty} c_n v^2 x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r}$$

$$= \sum_{n=2}^{\infty} \left[c_n (n+r)^2 - c_n v^2 + c_{n-2}\right] x^{n+r} + c_0 (r^2 - v^2) x^r + c_1 \left[(r+1)^2 - v^2\right] x^{r+1}$$

All these coefficients must be equal to zero!

$$c_0 (r^2 - v^2) = 0 \qquad c_1 [(r+1)^2 - v^2] = 0 \qquad c_n (n+r)^2 - c_n v^2 + c_{n-2} = 0$$

Since let $c_0 \neq 0$

$$c_0(r^2 - v^2) = 0 \quad \Rightarrow \quad r = v \quad \text{and} \quad r = -v$$

Let us first choose $r = v \geq 0$. Then, we have

$$c_1[(r+1)^2 - v^2] = 0 \quad \Rightarrow \quad c_1[(v+1)^2 - v^2] = (2v+1)c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$c_n(n+r)^2 - c_n v^2 + c_{n-2} = 0 \quad \Rightarrow \quad c_n = -\frac{1}{n(n+2v)} c_{n-2}, \; n = 2, 3, \cdots$$

$$\Downarrow$$

$$c_3 = -\frac{1}{3(3+2v)} c_1 = 0, \quad c_5 = -\frac{1}{5(5+2v)} c_3 = 0, \cdots$$

$$c_{2m+1} = 0, \quad m = 0, 1, 2, \cdots$$

Similarly,

$$c_2 = -\frac{1}{2(2+2v)}c_0 = \frac{-1}{2^2\cdot(1+v)}c_0$$

$$c_4 = -\frac{1}{4(4+2v)}c_2 = \frac{1}{4(4+2v)}\cdot\frac{1}{2(2+2v)}c_0 = \frac{1}{2^4\cdot 2\cdot(2+v)(1+v)}c_0$$

$$c_6 = -\frac{1}{6(6+2v)}c_4 = \frac{-1}{2^6\cdot 3\cdot 2\cdot(3+v)(2+v)(1+v)}c_0$$

$$\vdots$$

$$c_{2m} = -\frac{1}{2m(2m+2v)}c_{2m-2} = -\frac{1}{2^2 m(m+v)}c_{2m-2}$$

$$= \frac{1}{2^4 m(m-1)\cdot(m+v)(m-1+v)}c_{2m-4}$$

$$\vdots$$

$$= \frac{(-1)^m c_0}{2^{2m} m(m-1)\cdots 1\cdot(m+v)(m-1+v)\cdots(v+1)}, \quad m = 1, 2, \cdots$$

Thus, we obtain a solution to the Bessel′s equation of order v,

$$y_1(x) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m} m! \cdot (m+v)(m-1+v)\cdots(v+1)} x^{2m+v}$$

⇓

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n} n! \cdot (n+v)(n-1+v)\cdots(v+1)}$$

We need two linearly independent solutions to the Bessel′s equation in order to characterized all its solutions as Bessel′s equation is a 2nd order differential equation. We need to find another solution.

But, we will first introduce a **Gamma function** such that the above solution to the Bessel′s equation $y_1(x)$ can be re-written in a neater way.

Gamma Function

For $x > 0$, we define a so-called Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$

If $x > 0$, then $\Gamma(x+1) = x\,\Gamma(x)$.

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = -\int_0^\infty t^x de^{-t}$$

$$= -t^x e^{-t}\Big|_0^\infty + \int_0^\infty x t^{x-1} e^{-t} dt$$

$$= x\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x)$$

$$\lim_{t\to\infty} t^x e^{-t} = \lim_{t\to\infty} \frac{t^x}{e^t}, \quad k-1 \le x \le k$$

$$= \lim_{t\to\infty} \frac{x t^{x-1}}{e^t} = \cdots$$

$$= \lim_{t\to\infty} \frac{x\cdots(x-k+1)t^{x-k}}{e^t}$$

$$= \lim_{t\to\infty} \frac{x\cdots(x-k+1)}{t^{k-x}e^t} = 0$$

For any positive integer n,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)\cdots 1\cdot\Gamma(1)$$

$$= n!\int_0^\infty e^{-t} dt = n!\left(-e^{-t}\Big|_0^\infty\right) = n!\cdot 1 = n!$$

If $v \geq 0$, but v is not necessarily an integer,

$$\Gamma(n+v+1) = (n+v)\Gamma(n+v) = (n+v)(n+v-1)\Gamma(n+v-1)$$
$$= (n+v)(n+v-1)\cdots(v+1)\cdot\Gamma(v+1)$$

$$\Downarrow$$

$$(n+v)(n+v-1)\cdots(v+1) = \frac{\Gamma(n+v+1)}{\Gamma(v+1)}$$

This is known as the **factorial property** of the Gamma function.

Note that

$$\Gamma(x+1) = x\Gamma(x) \quad \Rightarrow \quad \Gamma(x) = \frac{1}{x}\Gamma(x+1)$$

This property holds for all $x > 0$. We will use the above property to define Gamma function for $x < 0$.

First we note that $\Gamma(x+1)$ is well defined for all $x > -1$. Thus, we can use

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1)$$

to define $\Gamma(x)$ for $-1 < x < 0$, i.e.,

$$\Gamma(-0.9) = \frac{1}{-0.9}\Gamma(-0.9+1) = -\frac{1}{0.9}\Gamma(0.1), \quad \Gamma(-0.3) = \frac{1}{-0.3}\Gamma(0.7)$$

Next, note that $\Gamma(x+1)$ is well defined for all $-2 < x < -1$. We use

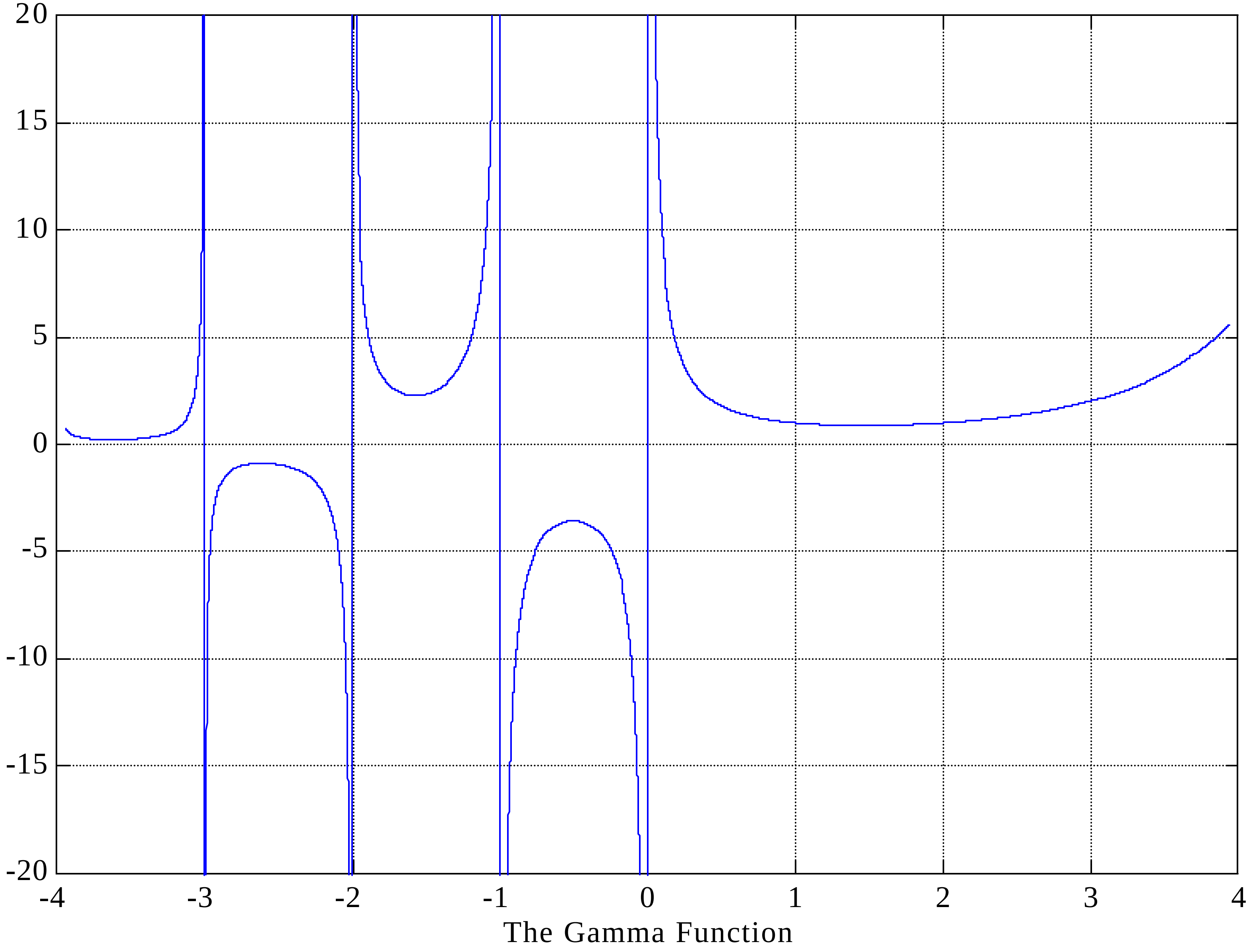
$$\Gamma(x) = \frac{1}{x}\Gamma(x+1)$$

to define $\Gamma(x)$ for $-2 < x < -1$, i.e.,

$$\Gamma(-1.8) = \frac{1}{-1.8}\Gamma(-1.8+1) = -\frac{1}{1.8}\Gamma(-0.8), \quad \Gamma(-1.2) = \frac{1}{-1.2}\Gamma(-0.2)$$

We can continue on this process forever to define $\Gamma(x)$ on every $(-n-1, -n)$.

The Gamma function



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Solution of Bessel's Equation in terms of Gamma Function

Recall that the first solution we have obtained for the Bessel′s equation, i.e.,

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n} \cdot n! \cdot (n+v)(n+v-1)\cdots(v+1)}$$

$$= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(v+1) x^{2n+v}}{2^{2n} \cdot n! \cdot \Gamma(n+v+1)} \qquad \hookleftarrow \text{ factorial property}$$

$$(n+v)(n+v-1)\cdots(v+1) = \frac{\Gamma(n+v+1)}{\Gamma(v+1)}$$

Let us choose

$$c_0 = \frac{1}{2^v \Gamma(v+1)} \quad \Longrightarrow \quad y_1(x) = J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} \cdot n! \cdot \Gamma(n+v+1)}$$

$J_v(x)$ is called a Bessel Function of the 1st kind of order v. This power series converges for all positive x (prove it yourself).

Exercise Problem: Verify that

$$J_v'(x) = \frac{1}{2}\left[J_{v-1}(x) - J_{v+1}(x)\right]$$

Solution:

$\Gamma(n+v+1) = (n+v)\,\Gamma(n+v)$

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

$$\Downarrow$$

$$J'_v(x) = \sum_{n=0}^{\infty} \frac{(2n+v)(-1)^n x^{2n+v-1}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

$$\Downarrow$$

$$J_{v-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v-1}}{2^{2n+v-1} n!\,\Gamma(n+v)} = \sum_{n=0}^{\infty} \frac{2(n+v)(-1)^n x^{2n+v-1}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

$$J_v'(x) - \frac{1}{2} J_{v-1}(x)$$

$$J'_v(x) - \frac{1}{2} J_{v-1}(x) = \sum_{n=0}^{\infty} \frac{n(-1)^n x^{2n+v-1}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+v-1}}{2^{2n+v} (n-1)!\,\Gamma(n+v+1)}$$

$n := n - 1$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+v+1}}{2^{2n+v+2} n!\,\Gamma(n+v+2)}$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v+1}}{2^{2n+v+1} n!\,\Gamma(n+v+2)}$$

$$= -\frac{1}{2} J_{v+1}(x)$$

$$\Rightarrow \quad \boxed{J'_v(x) = \frac{1}{2}\left[J_{v-1}(x) - J_{v+1}(x)\right]}$$

Second Solution to the Bessel′s Equation

We will now consider the problem of finding a second, linearly independent solution of Bessel′s equation. Recall that

$$F(r) = r^2 - v^2 = 0$$

which has two roots $r_1 = v$ and $r_2 = -v$.

Case 1 (Easy Case): v is not an integer.

Theorem: If v is not an integer, then two linearly independent solutions of Bessel′s equation of order v are

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n! \Gamma(n+v+1)} \quad \& \quad J_{-v}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-v}}{2^{2n-v} n! \Gamma(n-v+1)}$$

All other solutions can be expressed as linear combinations of these two, i.e.,

$$y(x) = a_1 J_v(x) + a_2 J_{-v}(x)$$

Case 2 (Complicated Case): v is a nonnegative integer

If v is a nonnegative integer, say $v = k$. In this case, $J_v(x)$ and $J_{-v}(x)$ are solutions of Bessel's equation of order v, but they are NOT linearly independent. This fact can be verified from the following arguments: First note that

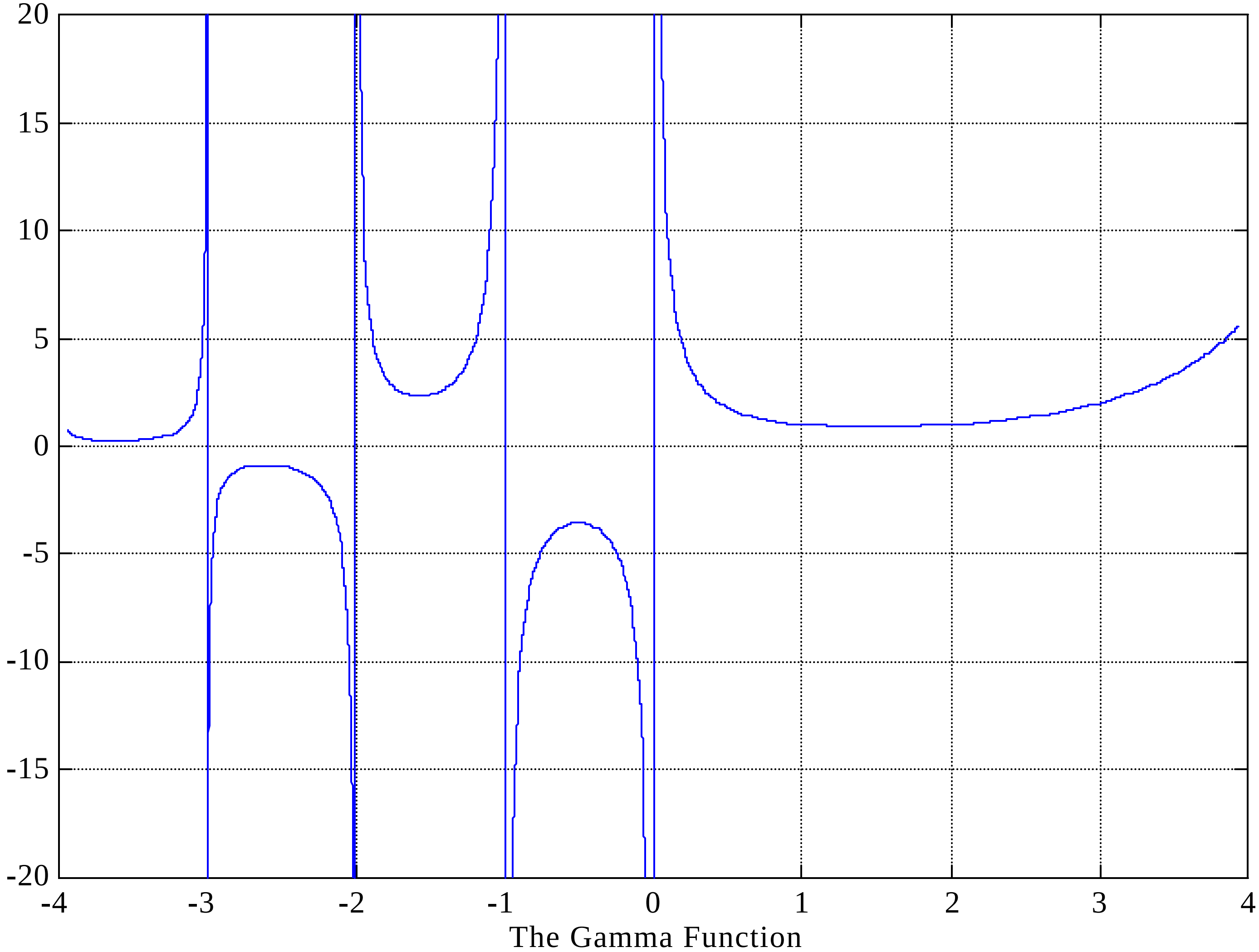
$$J_{-k}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-k}}{2^{2n-k} n! \Gamma(n-k+1)}$$

Observing the values of Gamma function at 0, –1, – 2, ..., they go to infinity. Thus we have

$$\Gamma(n-k+1) \to \infty \quad \text{or} \quad \frac{1}{\Gamma(n-k+1)} \to 0$$

for $n - k + 1 = 0, -1, -2, \ldots$, or $\mathbf{n = k - 1, k - 2, k - 3, \ldots, 0}$

The Gamma function



$$\Longrightarrow \quad J_{-k}(x)=\sum_{n=k}^{\infty}\frac{(-1)^n x^{2n-k}}{2^{2n-k}\, n!\,\Gamma(n-k+1)}$$

Changing the index ***n*** to ***m* + *k***, we obtain

$$J_{-k}(x)=\sum_{m=0}^{\infty}\frac{(-1)^{m+k} x^{2(m+k)-k}}{2^{2(m+k)-k}(m+k)!\,\Gamma(m+k-k+1)}$$

$$=\sum_{m=0}^{\infty}\frac{(-1)^{m}(-1)^{k} x^{2m+k}}{2^{2m+k}(m+k)!\,\Gamma(m+1)}$$

$$=(-1)^k\sum_{m=0}^{\infty}\frac{(-1)^{m} x^{2m+k}}{2^{2m+k}(m+k)!\,\Gamma(m+1)}=(-1)^k\sum_{m=0}^{\infty}\frac{(-1)^{m} x^{2m+k}}{2^{2m+k}(m+k)!\,m!}$$

$\Gamma(m+1) = m!$

$$=(-1)^k\sum_{m=0}^{\infty}\frac{(-1)^{m} x^{2m+k}}{2^{2m+k}\,\Gamma(m+k+1)\,m!}=(-1)^k J_k(x)$$

Thus, $J_k(x)$ and $J_{-k}(x)$ are linearly dependent.

Summary of results obtained so far:

We deal with Bessel′s equation in this topic,

$$x^2 y''+xy'+\left(x^2-v^2\right)y=0, \quad v\geq 0$$

We have shown that it has a solution,

$$J_v(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+v}}{2^{2n+v}n!\Gamma(n+v+1)}$$

If v is not an integer, we have $J_v(x)$ and $J_{-v}(x)$ being linearly independent.

Hence, all its solution can be expressed as,

$$y(x)=a_1 J_v(x)+a_2 J_{-v}(x)$$

If v is an integer, $J_v(x)$ and $J_{-v}(x)$ are linearly dependent and we need to search for a new solution …….

A 2nd Solution to Bessel's Equation for Case $v = k = 0$

Let us try a solution of the following format (why? Only God knows.)

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* \, x^n \quad \text{where} \quad y_1(x) = J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

$$\Rightarrow \quad y'_2(x) = J'_0(x)\ln(x) + \frac{1}{x}J_0(x) + \sum_{n=1}^{\infty} n c_n^* \, x^{n-1}$$

$$\Downarrow$$

$$y''_2(x) = J''_0(x)\ln(x) + \frac{2}{x}J'_0(x) - \frac{1}{x^2}J_0(x) + \sum_{n=1}^{\infty} n(n-1)\, c_n^* \, x^{n-2}$$

If $y_2(x)$ is a solution to the Bessel's equation of order 0, it must satisfy its differential equation….

Substituting the above equations into Bessel′s equation of order 0, i.e.,

$$x^2 y''+xy'+\left(x^2-0^2\right)y = xy''+y'+xy = 0$$

$$\Rightarrow$$

$$0 = x\,J''_0(x)\ln(x) + 2J'_0(x) - \frac{1}{x}J_0(x) + \sum_{n=1}^{\infty} n(n-1)c_n^*\,x^{n-1}$$

$$+ J'_0(x)\ln(x) + \frac{1}{x}J_0(x) + \sum_{n=1}^{\infty} n c_n^*\,x^{n-1}$$

$$+ x\,J_0(x)\ln(x) + \sum_{n=1}^{\infty} c_n^*\,x^{n+1}$$

$$\sum_{n=1}^{\infty} n^2 c_n^*\,x^{n-1}$$

$$0 = \ln(x)\left[x\,J''_0(x) + J'_0(x) + x\,J_0(x)\right]$$

$$2J'_0(x) + \sum_{n=1}^{\infty} n^2 c_n^*\,x^{n-1} + \sum_{n=1}^{\infty} c_n^*\,x^{n+1} = 0$$

Note that

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \Rightarrow J'_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-1} n!(n-1)!}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + \sum_{n=1}^{\infty} n^2 c_n^* x^{n-1} + \sum_{n=1}^{\infty} c_n^* x^{n+1} = 0$$

$n := m - 2$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + c_1^* + 4c_2^* x + \sum_{n=3}^{\infty} n^2 c_n^* x^{n-1} + \sum_{n=3}^{\infty} c_{n-2}^* x^{n-1} = 0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + c_1^* + 4c_2^* x + \sum_{n=3}^{\infty} \left[n^2 c_n^* + c_{n-2}^*\right] x^{n-1} = 0$$

This has only odd power terms.

$c_1^* = 0$

This has even power terms when $n = 3, 5, 7, \ldots$ Their associated coefficients = 0.

$$n^2 c_n^* + c_{n-2}^* = 0, \qquad n = 3,5,7,\cdots$$

$$\Downarrow$$

$$c_n^* = -\frac{1}{n^2} c_{n-2}^*, \qquad n = 3,5,7,\cdots$$

$$\Rightarrow \left\{ \begin{aligned} c^*_3 &= -\frac{1}{9} c^*_1 = 0 \\ c^*_5 &= -\frac{1}{25} c^*_3 = 0 \\ c^*_7 &= -\frac{1}{49} c^*_5 = 0 \cdots \end{aligned} \right.$$

$$\Rightarrow \quad c^*_{2m+1} = 0 \quad \text{for} \quad m = 0, 1, 2 \cdots$$

We will now determine the remaining coefficients. First we replace n by $2j$ in the second summation and $n=j$ in the first summation in the following eq.

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + c_1^* + 4c_2^* x + \sum_{n=3}^{\infty} \left[\underbrace{n^2 c_n^* + c_{n-2}^*}_{=0 \text{ when } n=3} \right] x^{n-1} = 0$$

$$\Downarrow$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j x^{2j-1}}{2^{2j-2} j!(j-1)!} + 4c^*_2 \, x + \sum_{j=2}^{\infty} \left[4j^2 c^*_{2j} + c^*_{2j-2} \right] x^{2j-1} = 0$$

$$\Downarrow$$

$$(4c^*_2 - 1)x + \sum_{j=2}^{\infty} \left[\frac{(-1)^j}{2^{2j-2} j!(j-1)!} + 4j^2 c^*_{2j} + c^*_{2j-2} \right] x^{2j-1} = 0$$

$$\Longrightarrow \quad c^*_2 = \frac{1}{4}, \quad \text{and} \quad c^*_{2j} = \frac{(-1)^{j+1}}{2^{2j} j^2 j!(j-1)!} - \frac{1}{4j^2} c^*_{2j-2}$$

$$\Longrightarrow \quad c^*_4 = \frac{-1}{2^4 2^2 2} - \frac{1}{2^2 2^2 4} = \frac{-1}{2^2 4^2}\left[1 + \frac{1}{2}\right]$$

$$\Longrightarrow \quad c^*_6 = \frac{1}{2^6 3^2 6 \cdot 2} + \frac{1 + \frac{1}{2}}{4 \cdot 3^2 2^2 4^2} = \frac{1}{2^2 4^2 6^2}\left[1 + \frac{1}{2} + \frac{1}{3}\right]$$

$$\Longrightarrow \quad c^*_{2j} = \frac{(-1)^{j+1}}{2^2 4^2 \cdots (2j)^2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}\right] = \frac{(-1)^{j+1}}{2^{2j}(j!)^2} y(j)$$

$$y(j) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$$

A second solution of Bessel′s equation of order zero may be written as

$$y_2(x) = J_0(x)\ln(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}(n!)^2} y\,(n) x^{2n}, \quad x > 0$$

Because of the logarithm term, this second solution is linearly independent from the first solution, $J_0(x)$.

Instead of using $y_2(x)$ for a second solution, it is customary to use a particular linear combination of $J_0(x)$ and $y_2(x)$, denoted $Y_0(x)$ and defined by

$$Y_0(x) = \frac{2}{p}\{\, y_2(x) + [g - \ln(2)] J_0(x) \,\}$$

where γ is called Euler′s constant and is given by

$$g = \lim_{n\to\infty}[y\,(n) - \ln(n)] = 0.577215664901533\cdots$$

Since $Y_0(x)$ is a linear combination of the solutions of Bessel′s equation of order 0, i.e., $J_0(x)$ and $y_2(x)$, it is also a solution. Furthermore, $Y_0(x)$ is linearly independent from $J_0(x)$. Thus, the general solution of Bessel′s equation of order 0 is given by

$$y(x) = a_1 J_0(x) + a_2 Y_0(x)$$

In view of the series derived above for $y_2(x)$,

$$\begin{aligned} Y_0(x) &= \frac{2}{p}\left\{ J_0(x)\ln(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}(n!)^2} y(n) x^{2n} + [g - \ln(2)] J_0(x) \right\} \\ &= \frac{2}{p}\left\{ J_0(x)\left[\ln\left(\frac{x}{2}\right) + g\right] + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}(n!)^2} y(n) x^{2n} \right\} \end{aligned}$$

$Y_0(x)$ is a Bessel′s function of the 2nd kind of order zero. With the above choice of constants, $Y_0(x)$ is also called Neumann′s function of order 0.

A 2nd Solution of Bessel′s Equation of Order v (positive integer).

If v is a positive integer, say $v = k$, then a similar procedure as in the $k = 0$ case, but more involved calculation leads us to the following 2nd solution of Bessel′s equation of order $v = k$,

$$Y_k(x) = \frac{2}{p}\left\{ J_k(x)\left[\ln\left(\frac{x}{2}\right) + g\right] - \sum_{n=0}^{k-1} \frac{(k-n-1)!}{2^{2n-k+1} n!} x^{2n-k} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}[y(n) + y(n+k)]}{2^{2n+k+1} n!(n+k)!} x^{2n+k} \right\}$$

$Y_k(x)$ and $J_k(x)$ are linearly independent for $x > 0$, and the general solution of Bessel′s equation of order k is given by

$$y(x) = a_1 J_k(x) + a_2 Y_k(x)$$

Although $J_k(x)$ is simple $J_v(x)$ for the case $v = k$, our derivation of $Y_k(x)$ does not suggest how $Y_v(x)$ might be defined if v is not a nonnegative integer.

However, it is possible to define $Y_v(x)$, if v is not an integer, by letting

$$Y_v(x) = \frac{1}{\sin(v\pi)}\left[J_v(x)\cos(v\pi) - J_{-v}(x)\right]$$

This is a linear combination of $J_v(x)$ and $J_{-v}(x)$, two solutions of Bessel′s equation of order v, and hence is also a solution of Bessel′s equation of order v.

It can be shown (very complicated!) that one can obtain $Y_k(x)$, for k being a non-negative integer, from the above definition by taking the limit,

$$Y_k(x) = \lim_{v \to k} Y_v(x)$$

$Y_v(x)$ is called Neumann′s Bessel function of order v. It is linearly independent from $J_v(x)$ and hence the general solution of Bessel′s equation of order v (regardless it is an integer or not) can be written as

$$y(x) = a_1 J_v(x) + a_2 Y_v(x)$$

Extra: Linear Dependence and Linear Independence

Given two functions $f(x)$ and $g(x)$, they are said to be linearly independent if and only if

$$\boxed{a\cdot f(x)+b\cdot g(x)=0 \quad \text{for all } x \text{ defined}}$$

holds with $a=0$ and $b=0$. Otherwise, they are said to be dependent, i.e., there exist either nonzero a and/or nonzero b such that

$$a\cdot f(x)+b\cdot g(x)=0 \quad \text{for all } x \text{ defined.}$$

Assume that a is nonzero. We can then rewrite the above equation as

$$f(x)=\left(-b/a\right)\cdot g(x)=\boldsymbol{a}\cdot g(x)$$

$f(x)$ and $g(x)$ are related by a constant and hence they are dependent to one another.

Now, given $J_v(x)$ and $J_{-v}(x)$ are linearly independent, show that $J_v(x)$ and $Y_v(x)$ with

$$Y_v(x) = \frac{1}{\sin(vp)}\left[J_v(x)\cos(vp) - J_{-v}(x)\right]$$

are also linearly independent.

Proof. Rewrite $Y_v(x)$ as

$$Y_v(x) = \frac{1}{\sin(vp)}\left[J_v(x)\cos(vp) - J_{-v}(x)\right] = \boldsymbol{a}\, J_v(x) + \boldsymbol{b}\, J_{-v}(x), \qquad \boldsymbol{b} \neq 0$$

and let

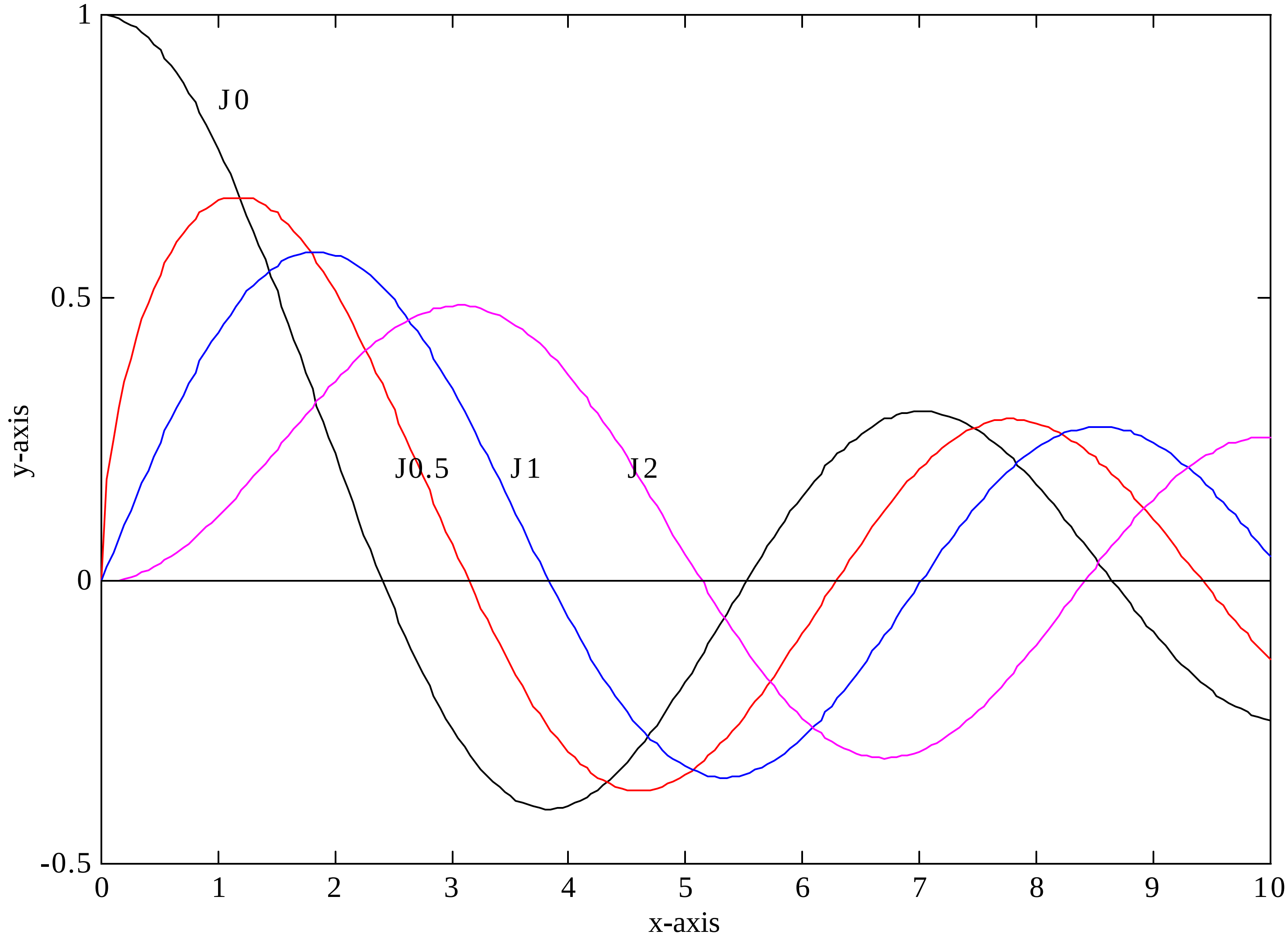
$$a \cdot J_v(x) + b \cdot Y_v(x) = 0 \quad \Rightarrow \quad a\, J_v(x) + b\left[\boldsymbol{a}\, J_v(x) + \boldsymbol{b}\, J_{-v}(x)\right] = 0$$

$$\Rightarrow \quad (a + b\boldsymbol{a})\, J_v(x) + b\boldsymbol{b}\, J_{-v}(x) = 0$$

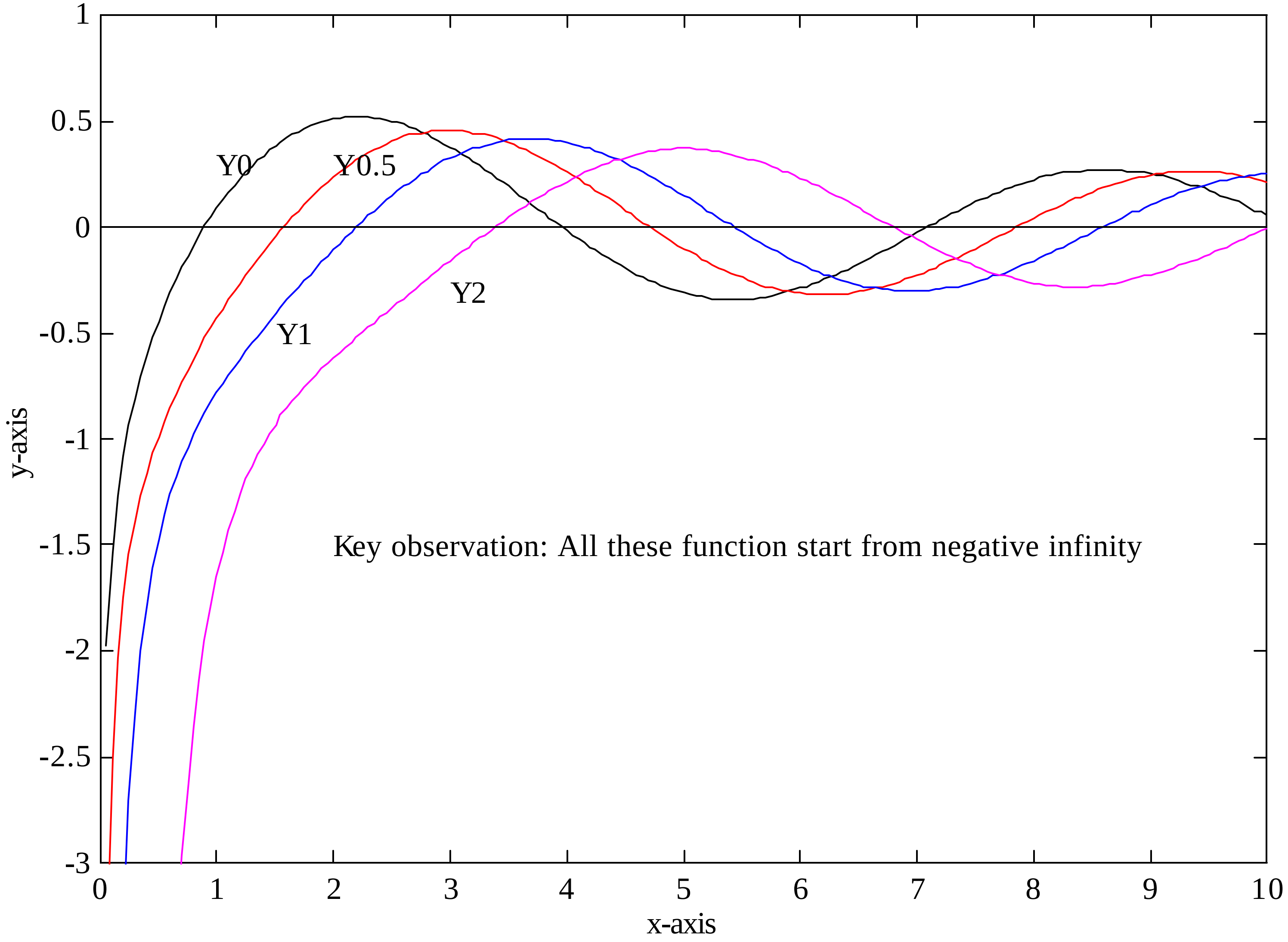
$$\Rightarrow \quad (a + b\boldsymbol{a}) = 0, \qquad b\boldsymbol{b} = 0 \qquad \Rightarrow \qquad b = 0, \; a = 0$$

Hence, $J_v(x)$ and $Y_v(x)$ are linearly independent.

The Bessel functions of 1st kind



Bessel functions of the 2nd kind



Modified Bessel Functions

Sometimes, modified Bessel functions are encountered in modeling physical phenomena. First, we can show that

$$y(x) = a_1 J_0(kx) + a_2 Y_0(kx)$$

is the general solution of the following differential equation

$$y'' + \frac{1}{x} y' + k^2 y = 0$$

Proof. We prove this for $y(x) = J_0(kx)$ only. The rest can similarly be shown.

$$y'' + \frac{1}{x} y' + k^2 y = k^2 J_0''(kx) + \frac{1}{x} k J_0'(kx) + k^2 J_0(kx)$$

$$= \frac{k}{x}[(kx)J_0''(kx) + J_0'(kx) + (kx)J_0(kx)] = \frac{k^2}{z}[zJ_0''(z) + J_0'(z) + zJ_0(z)] = 0$$

Now, let $k = i$, where $i = \sqrt{-1}$, which implies $k^2 = i^2 = -1$. Then

$$y(x) = a_1 J_0(ix) + a_2 Y_0(ix)$$

is the general solution of

$$y'' + \frac{1}{x} y' - y = 0$$

which is called a modified Bessel's equation of order 0, and $J_0(ix)$ is a modified Bessel function of the first kind of order 0. Usually, we denote

$$I_0(x) = J_0(ix)$$

Since $i^2 = -1$, substitution of ix for x in the series for J_0 yields:

$$I_0(x) = 1 + \frac{1}{2^2} x^2 + \frac{1}{2^2 4^2} x^4 + \frac{1}{2^2 4^2 6^2} x^6 + \cdots$$

It is a real function of x.

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

Usually $Y_0(ix)$ is not used. Instead we use the function

$$K_0(x) = [\ln(2) - g]I_0(x) - I_0(x)\ln(x) + \frac{1}{4}x^2 + \cdots$$

$K_0(x)$ is called a modified Bessel function of the second kind of order zero.

We now write the general solution of the differential equation

$$y'' + \frac{1}{x}y' - y = 0$$

as

$$y(x) = a_1 I_0(x) + a_2 K_0(x)$$

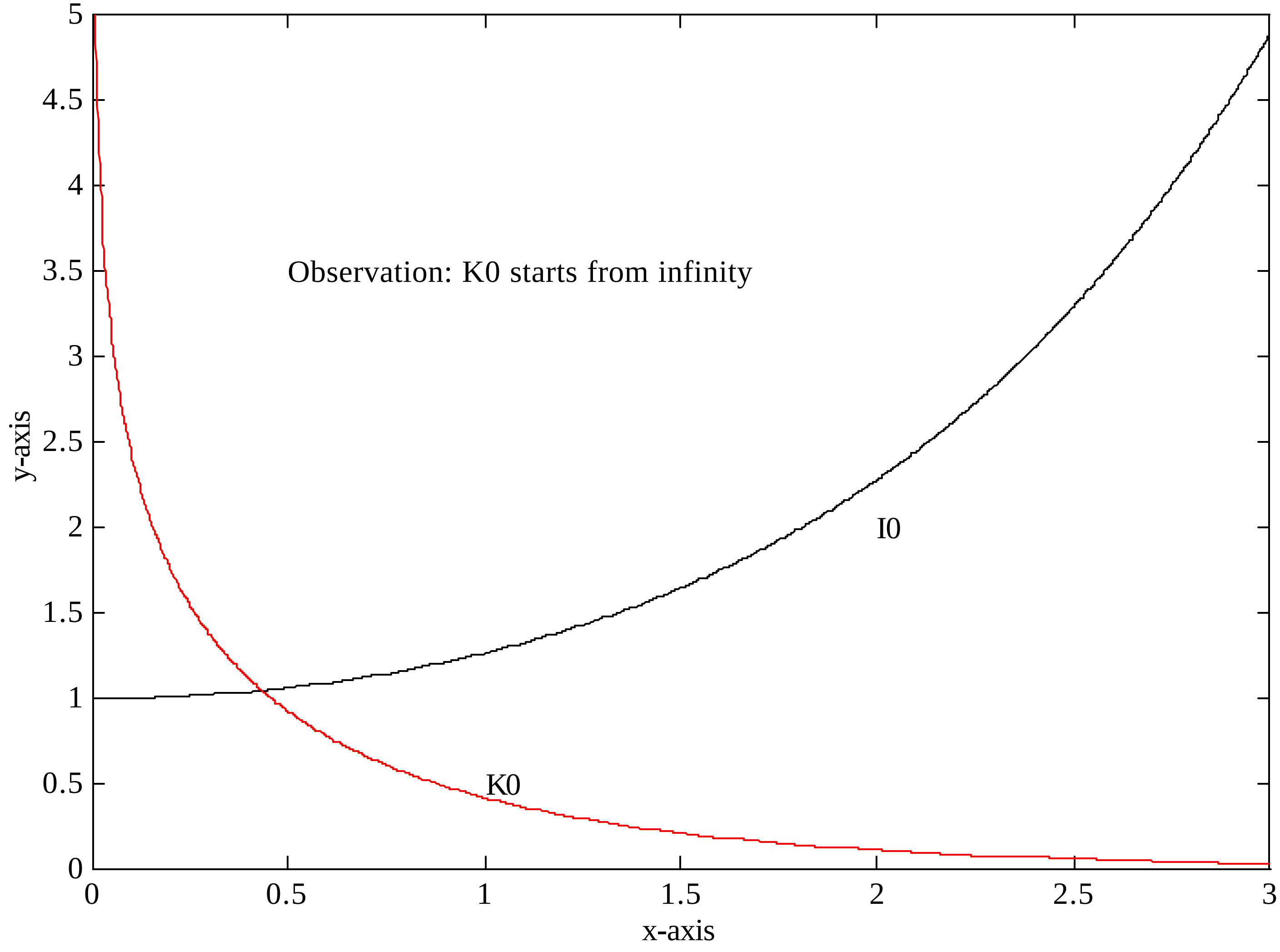
Homework: Show that the general solution of the differential equation

$$y'' + \frac{1}{x}y' - b^2 y = 0$$

is given by

$$y(x) = a_1 I_0(bx) + a_2 K_0(bx)$$

The modified Bessel functions of order 0



Exercise Problem: (Problem 21, O′Neil, page 262) Show that

$$[xI_0'(x)]' = xI_0(x)$$

Proof.

$$I_0(x) = J_0(ix)$$
$$\Downarrow$$
$$I_0'(x) = iJ_0'(ix)$$
$$\Downarrow$$
$$xI_0'(x) = ixJ_0'(ix)$$
$$\Downarrow$$
$$[xI_0'(x)]' = iJ_0'(ix) - xJ_0''(ix)$$

Note that $y = J_0(ix)$ is the sln of the modified Bessel′s equation of order 0

$$xy'' + y' = xy$$

and

$$y = J_0(ix) \Rightarrow y' = iJ_0'(ix)$$
$$\Rightarrow y'' = -J_0''(ix)$$

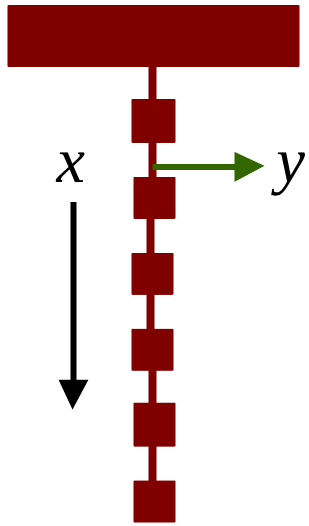
Hence

$$xJ_0(ix) = iJ_0'(ix) - xJ_0''(ix)$$

$$[xI_0'(x)]' = xI_0(x)$$

Applications of Bessel Functions (Oscillations of a Suspended Chain)

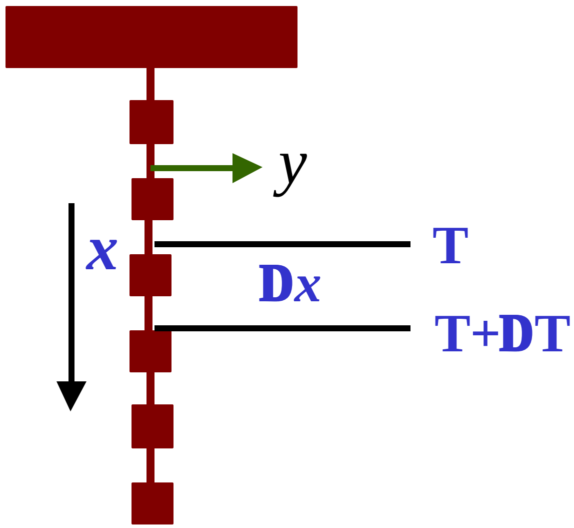
Displacement of a Suspended Chain

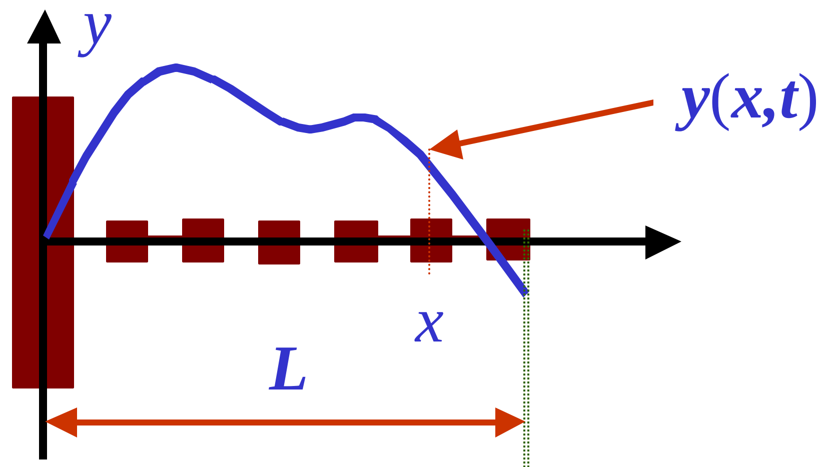


Suppose we have a heavy flexible chain. The chain is fixed at the upper end and free at the bottom.

We want to describe the oscillations caused by a small displacement in a horizontal direction from the stable equilibrium position.

Assume that each particle of the chain oscillates in a horizontal straight line. Let ***m*** be the mass of the chain per unit length, ***L*** be the length of the chain, and ***y***(***x***,***t***) be the horizontal displacement at time ***t*** of the particle of chain whose distance from the point of suspension is ***x***.





Consider an element of chain of length **Dx**. The forces acting on the ends of this element are **T** and **T+DT**, the horizontal component in Newton's 2nd Law of Motion (force equals to the rate of change of momentum with respect to time) is:

$$F = ma \quad \Rightarrow \quad m(\cancel{\Delta x})\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)\cancel{\Delta x}$$

$$\Longrightarrow \quad m\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)$$

where $T = mg(L - x)$ is the weight of the chain below where **T** acts

$$\Downarrow$$

$$m\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(mgL\frac{\partial y}{\partial x} - mgx\frac{\partial y}{\partial x}\right) = mgL\frac{\partial^2 y}{\partial x^2} - mg\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right)$$

$$= mgL\frac{\partial^2 y}{\partial x^2} - mgx\frac{\partial^2 y}{\partial x^2} - mg\frac{\partial y}{\partial x} = m\left(-g\frac{\partial y}{\partial x} + g(L-x)\frac{\partial^2 y}{\partial x^2}\right)$$

$$\Longrightarrow \quad \frac{\partial^2 y}{\partial t^2} = -g\frac{\partial y}{\partial x} + g(L-x)\frac{\partial^2 y}{\partial x^2}$$

This is a partial differential equation. However, we can reduce it to a problem involving only an ordinary differential equation.

Let $z = L - x$ and $u(z, t) = y\,(L - z, t)$. Then

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} \qquad \frac{\partial y}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = -\frac{\partial u}{\partial z}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = -\frac{\partial}{\partial z}\left(\frac{\partial y}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\frac{\partial x}{\partial z} = \frac{\partial^2 y}{\partial x^2}$$

$$\frac{\partial^2 y}{\partial t^2} = -g\,\frac{\partial y}{\partial x} + g(L - x)\frac{\partial^2 y}{\partial x^2} \quad \Rightarrow \quad \frac{\partial^2 u}{\partial t^2} = g\,\frac{\partial u}{\partial z} + gz\,\frac{\partial^2 u}{\partial z^2}$$

This is still a partial differential equation, which can be solved using p.d.e. method. Since we anticipate the oscillations to be periodic in t, we will attempt a solution of the form $u(z,t) = f(z)\cos(wt - q)$

$$-w^2 f(z)\cos(wt - q) = gf'(z)\cos(wt - q) + gzf''(z)\cos(wt - q)$$

Dividing the above equation by $gz\cos(wt-d)$, we get a differential equation

$$f''(z)+\frac{1}{z}f'(z)+\frac{w^2}{gz}f(z)=0$$

It is shown in the lecture notes that $x^a J_n(bx^c)$ and $x^a Y_n(bx^c)$ are the solutions of

$$y''-\left(\frac{2a-1}{x}\right)y'+\left(b^2c^2x^{2c-2}+\frac{a^2-n^2c^2}{x^2}\right)y=0$$

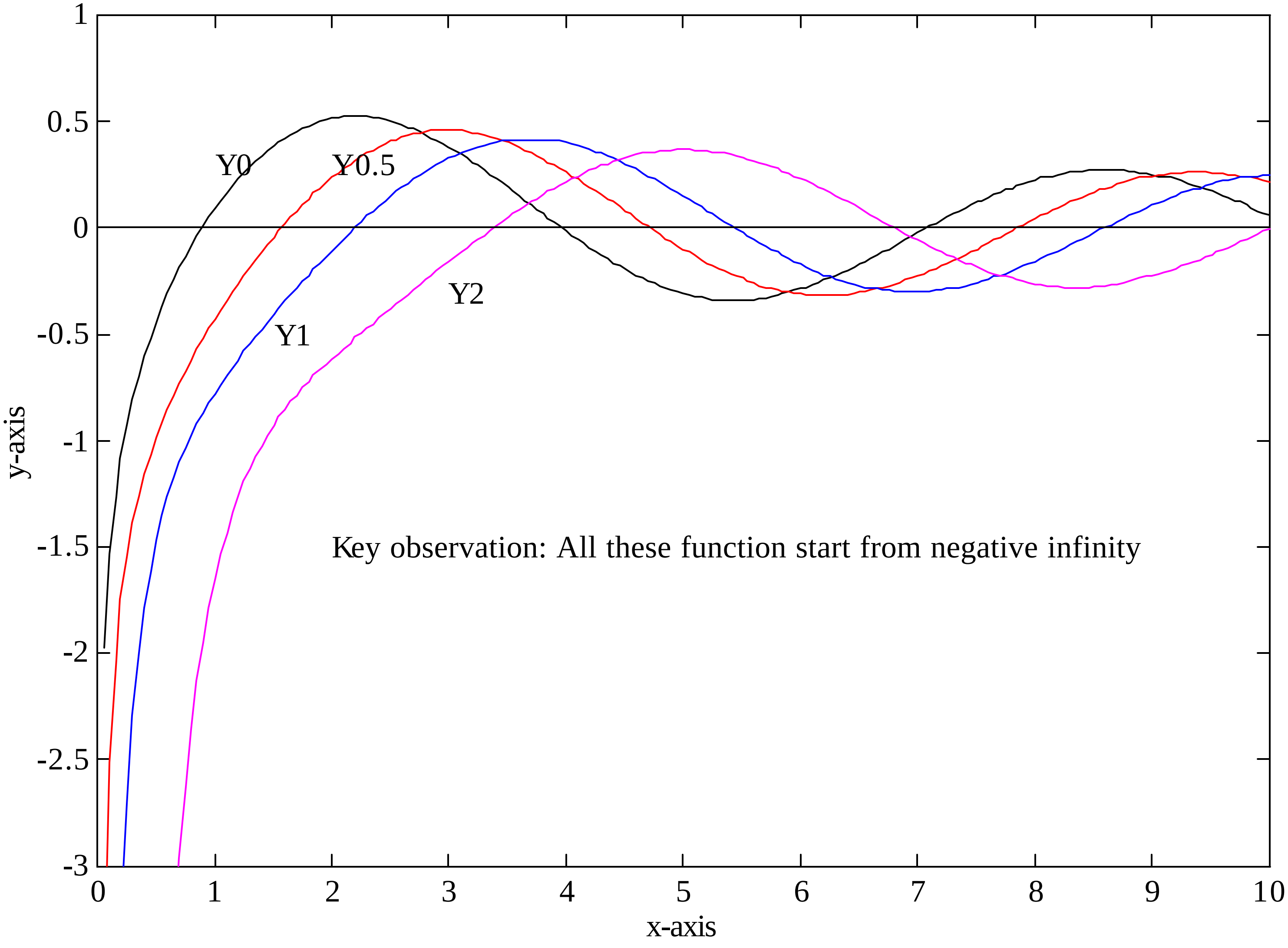
We let

$$\left.\begin{aligned} 2a-1=-1 &\quad\Rightarrow\quad a=0 \\ 2c-2=-1 &\quad\Rightarrow\quad c=\frac{1}{2} \\ b^2c^2=\frac{w^2}{g} &\quad\Rightarrow\quad b=\frac{2w}{\sqrt{g}} \\ a^2-n^2c^2=0 &\quad\Rightarrow\quad n=0 \end{aligned}\right\}$$

Thus, the general solution is in terms of Bessel functions of order zero:

$$f(z)=a_1J_0\left(2w\sqrt{\frac{z}{g}}\right)+a_2Y_0\left(2w\sqrt{\frac{z}{g}}\right)$$

Bessel functions of the 2nd kind



$$Y_0\left(2w\sqrt{\frac{z}{g}}\right) \to -\infty, \qquad \text{as } z \to 0 \quad \text{or} \quad x \to L$$

We must therefore choose $\alpha_2 = 0$ in order to have a bounded solution, as expected from the physical setting of the problem. This leaves us with

$$f(z) = a_1 J_0\left(2w\sqrt{\frac{z}{g}}\right)$$

Thus,

$$u(z,t) = f(z)\cos(wt - q) = a_1 J_0\left(2w\sqrt{\frac{z}{g}}\right)\cos(wt - q)$$

and hence

$$\boxed{y(x,t) = a_1 J_0\left(2w\sqrt{\frac{L-x}{g}}\right)\cos(wt - q)}$$

The frequencies of the normal oscillations of the chain are determined by using this general form of the solution for $y(x,t)$ together with condition that the upper end of the chain is fixed and therefore does not move. For all t, we must have

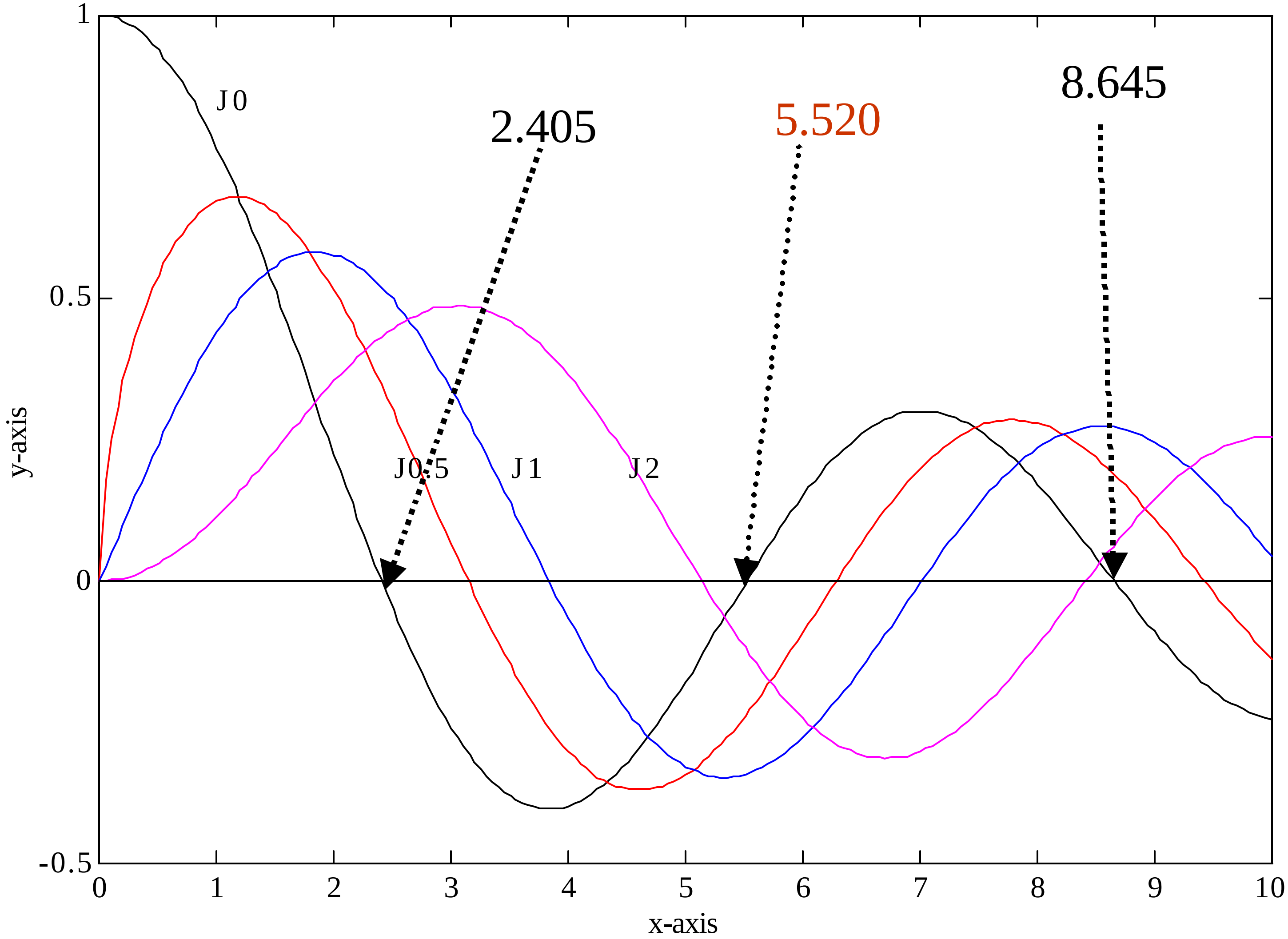
$$y(0,t) = a_1 J_0\left(2w\sqrt{\frac{L}{g}}\right)\cos(wt - q) = [a_1 \cos(wt - q)]J_0\left(2w\sqrt{\frac{L}{g}}\right) = 0$$

$$\Downarrow$$

$$J_0\left(2w\sqrt{\frac{L}{g}}\right) = 0$$

This gives values of ω which are frequencies of the oscillations. To find these admissible values of ω, we must consult a table of zeros of J_0. From a table of values of zeros of Bessel functions, we find that the first five positive solutions of $J_0(\mathbf{a}) = 0$ are approximately $\mathbf{a} = 2.405, 5.520, 8.645, 11.792, 14.931.$

The Bessel functions of 1st kind



Using the these zeros, we obtain

$$2w_1\sqrt{\frac{L}{g}} = 2.405 \qquad \Rightarrow \qquad w_1 = 1.203\sqrt{\frac{g}{L}}$$

$$2w_2\sqrt{\frac{L}{g}} = 5.520 \qquad \Rightarrow \qquad w_2 = 2.760\sqrt{\frac{g}{L}}$$

$$2w_3\sqrt{\frac{L}{g}} = 8.645 \qquad \Rightarrow \qquad w_3 = 4.327\sqrt{\frac{g}{L}}$$

$$2w_4\sqrt{\frac{L}{g}} = 11.792 \qquad \Rightarrow \qquad w_4 = 5.896\sqrt{\frac{g}{L}}$$

$$2w_5\sqrt{\frac{L}{g}} = 14.931 \qquad \Rightarrow \qquad w_5 = 7.466\sqrt{\frac{g}{L}}$$

All these are admissible values of w, and they represent frequencies of the normal modes of oscillation. The period T_j associated with w_j is

$$T_j = \frac{2p}{w_j}$$

Legendre′s Equation and Legendre Polynomials

The following 2nd order linear differential equation

$$\left(1-x^2\right)y''-2xy'+\alpha\left(\alpha+1\right)y=0$$

where α is a constant, is called **Legendre′s Equation**. It occurs in a variety of problems involving quantum mechanics, astronomy and analysis of heat conduction, and is often seen in settings in which it is natural to use spherical coordinates. We can also re-write Legendre′s equation as

$$y''-\left\{\frac{2x}{1-x^2}\right\}y'+\left\{\frac{\alpha\left(\alpha+1\right)}{1-x^2}\right\}y=0$$

The coefficient functions are analytic at every point except $x = 1$ and $x = -1$. In particular, both functions have series expansions in **(-1, 1).**

Thus, in general, we are only interested in finding solutions to Legendre′s equation in the interval **(-1, 1).** Since Legendre′s equation is a 2nd order differential equation, we will have to find two linearly independent solutions in order to characterize all its solutions. As in the Bessel′s equation case, we will try to find the solutions of Legendre′s equation in terms of power series. Since 0 is a solution, we let

$$y(x)=\sum_{n=0}^{\infty} a_n x^n \;\Rightarrow\; y'(x)=\sum_{n=0}^{\infty} n a_n x^{n-1}=\sum_{n=1}^{\infty} n a_n x^{n-1} \;\Rightarrow\; y''(x)=\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$+$$

$$\left(1-x^2\right)y''-2xy'+a\left(a+1\right)y=0$$

$$\Downarrow$$

shift up power by 2

$$\sum_{n=2}^{\infty} n\left(n-1\right)a_n x^{n-2}-\sum_{n=2}^{\infty} n\left(n-1\right)a_n x^{n}-\sum_{n=1}^{\infty} 2n a_n x^{n}+\sum_{n=0}^{\infty} a\left(a+1\right)a_n x^{n}=0$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty}n(n-1)a_n x^n - \sum_{n=1}^{\infty}2na_n x^n + \sum_{n=0}^{\infty}\alpha(\alpha+1)a_n x^n = 0$$

$$[2a_2 + \alpha(\alpha+1)a_0] + [6a_3 - 2a_1 + \alpha(\alpha+1)a_1]x$$

$$+\sum_{n=2}^{\infty}\left\{(n+2)(n+1)a_{n+2} - [n^2 + n - \alpha(\alpha+1)]a_n\right\}x^n = 0$$

$$2a_2 + \alpha(\alpha+1)a_0 = 0 \Rightarrow a_2 = -\frac{\alpha(\alpha+1)}{2}a_0$$

$$6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0 \Rightarrow a_3 = -\frac{(\alpha-1)(\alpha+2)}{6}a_1$$

$$(n+2)(n+1)a_{n+2} - [n^2 + n - \alpha(\alpha+1)]a_n = 0$$

$$\| $$

$$(n-\alpha)(n+\alpha+1)$$

$$a_{n+2} = -\frac{(\alpha-n)(n+\alpha+1)}{(n+2)(n+1)}a_n$$

For the even-indexed coefficients, we have

$$a_2 = -\frac{(a+1)a}{1\cdot 2}a_0$$

$$a_4 = -\frac{(a+3)(a-2)}{3\cdot 4}a_2 = +\frac{(a+3)(a+1)a(a-2)}{4!}a_0$$

$$a_6 = -\frac{(a+5)(a+3)(a+1)a(a-2)(a-4)}{6!}a_0$$

$$\vdots$$

$$a_{2n} = (-1)^n \frac{(a+2n-1)(a+2n-3)\cdots(a+1)a(a-2)\cdots(a-2n+2)}{(2n)!}a_0$$

Similarly, for the odd-indexed coefficients, we have

$$a_{2n+1} = (-1)^n \frac{(a+2n)(a+2n-2)\cdots(a+2)(a-1)(a-3)\cdots(a-2n+1)}{(2n+1)!}a_1$$

We can obtain two linearly independent solutions of Legendre′s equation by making choice for a_0 and a_1.

If we choose $a_0 = 1$ and $a_1 = 0$, we get one solution:

$$y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(a+2n-1)\cdots(a+1)a(a-2)\cdots(a-2n+2)}{(2n)!} x^{2n}$$
$$= 1 - \frac{(a+1)a}{2} x^2 + \frac{(a+3)(a+1)a(a-2)}{24} x^4 - \frac{(a+5)(a+3)(a+1)a(a-2)(a-4)}{720} x^6 + \cdots$$

If we choose $a_0 = 0$ and $a_1 = 1$, we get another solution:

$$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{(a+2n)\cdots(a+2)(a-1)(a-3)\cdots(a-2n+1)}{(2n+1)!} x^{2n+1}$$
$$= x - \frac{(a+2)(a-1)}{6} x^3 + \frac{(a+4)(a+2)(a-1)(a-3)}{120} x^5 - \cdots$$

Any solution to the Legendre′s equation can be expressed as a linear combination of the above two solutions.

Exercise Problem: Show that these solutions converge absolutely in (–1, 1).

Recall that the first solution,

$$y_1(x) = 1 - \frac{(a+1)a}{2}x^2 + \frac{(a+3)(a+1)a(a-2)}{24}x^4 - \frac{(a+5)(a+3)(a+1)a(a-2)(a-4)}{720}x^6 + \cdots$$

When $\alpha = 0$, we have

$$y_1(x) = 1$$

When $\alpha = 2$, we have

$$y_1(x) = 1 - \frac{(2+1)2}{2}x^2 = 1 - 3x^2$$

When $\alpha = 4$, we have

$$y_1(x) = 1 - \frac{(4+1)4}{2}x^2 + \frac{(4+3)(4+1)4(4-2)}{24}x^4 = 1 - 10x^2 + \frac{35}{3}x^4$$

Note that all the above solutions are polynomials, this process can be carried on for any even integer α.

Similarly, one can obtain polynomial solutions for odd integer α from

$$y_2(x) = x - \frac{(a+2)(a-1)}{6}x^3 + \frac{(a+4)(a+2)(a-1)(a-3)}{120}x^5 - \cdots$$

$$\Downarrow$$

$$y_2(x) = x \qquad a = 1$$

$$y_2(x) = x - \frac{(3+2)(3-1)}{6}x^3 = x - \frac{5}{3}x^3 \qquad a = 3$$

$$\vdots$$

In fact, whenever α is a nonnegative integer, the power series for either $y_1(x)$ (if α is even) or $y_2(x)$ (if α is odd) reduces to a finite series, and we obtain a polynomial solution of Legendre′s equation. Such polynomial solutions are useful in many applications, including methods for approximating solutions of equations $f(x) = 0$.

In many applications, it is helpful to standardize specific polynomial solutions so that their values can be tabulated. The convention is to multiply $y_1(x)$ or $y_2(x)$ for each term by a constant which makes the value of the polynomial equal to **1** at $\boldsymbol{x = 1}$.

The resulting polynomials are called Legendre polynomials and are denoted by $P_n(x)$, i.e., $P_n(x)$ is the solution of Legendre′s equation with $a = n$. Here are the first six Legendre polynomials:

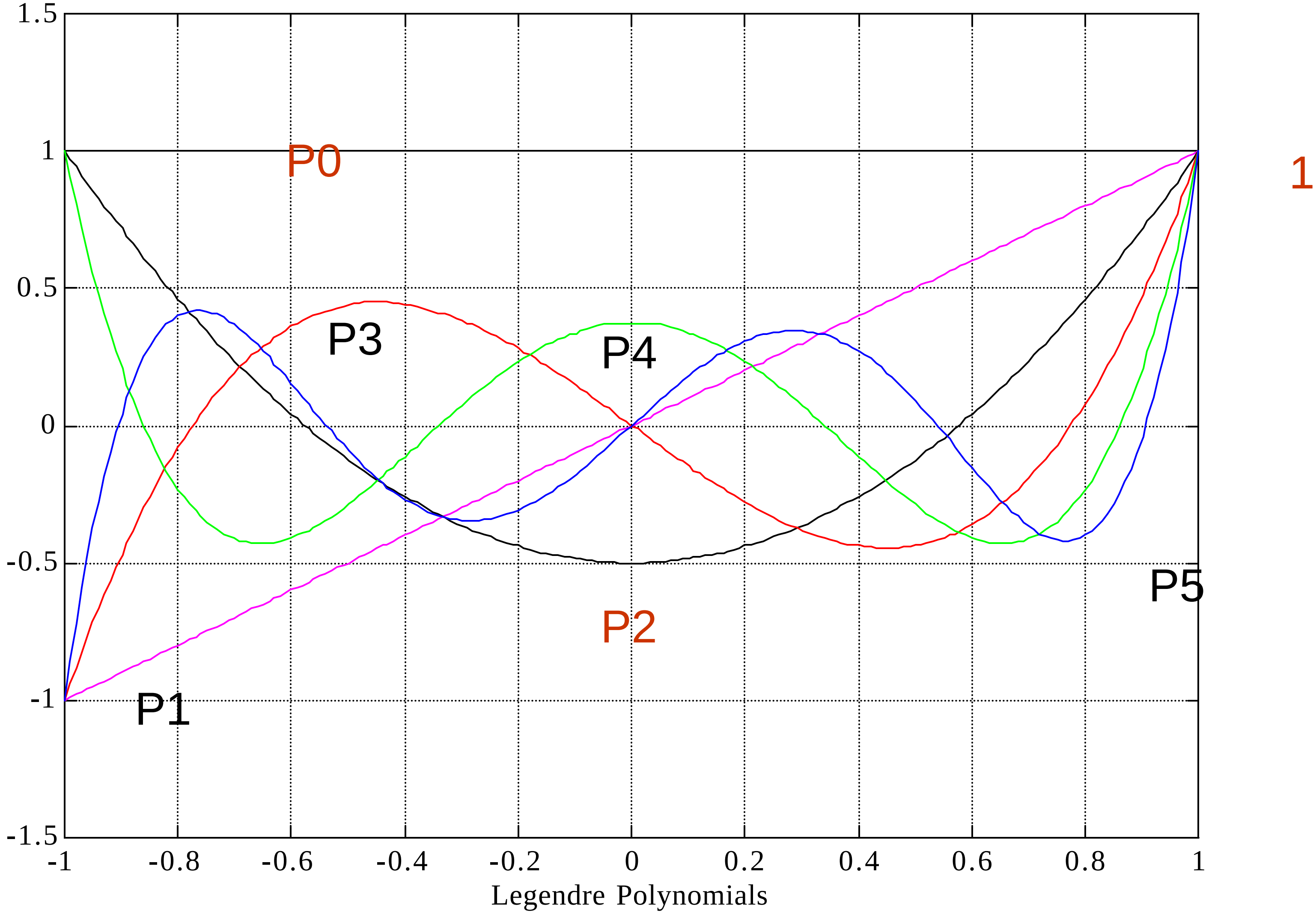
$$P_0(x) = 1, \qquad P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right), \qquad P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

Although these polynomials are defined for all x, they are solutions of Legendre′s equation only for $-1 < x < 1$ and for appropriate α.

Legendre Polynomials



Properties of Legendre Polynomials

Theorem 0. If ***m*** and ***n*** are distinct nonnegative integers, $\int_{-1}^{1} P_m(x)P_n(x)dx = 0$

Proof: Note that P_n and P_m are solutions of Legendre's differential equations with $a = n$ and $a = m$, respectively. Hence, we have

$$0 = (1-x^2)P_n'' - 2xP_n' + n(n+1)P_n \quad \times P_m$$

$$0 = (1-x^2)P_m'' - 2xP_m' + m(m+1)P_m \quad \times P_n$$

subtract these two equations –

$$0 = (1-x^2)P_n''P_m - 2xP_n'P_m + n(n+1)P_nP_m$$

$$0 = (1-x^2)P_m''P_n - 2xP_m'P_n + m(m+1)P_mP_n$$

$$= (1-x^2)(P_n''P_m - P_m''P_n) - 2x(P_n'P_m - P_m'P_n) + [n(n+1) - m(m+1)]P_mP_n = 0$$

$$= P_n''P_m + P_n'P_m' - P_m'P_n' - P_m''P_n$$

$$\Rightarrow (1-x^2)\frac{d}{dx}[P_n'P_m - P_m'P_n] - 2x[P_n'P_m - P_m'P_n] = [m(m+1) - n(n+1)]P_mP_n$$

$$\left(1-x^2\right)\frac{d}{dx}\left[P_n'P_m - P_m'P_n\right] - 2x\left[P_n'P_m - P_m'P_n\right] = \left[m(m+1) - n(n+1)\right]P_mP_n$$

$$\Downarrow$$

$$\frac{d}{dx}\left[\left(1-x^2\right)\left(P_n'P_m - P_m'P\right)\right] = \left[m(m+1) - n(n+1)\right]P_mP_n$$

$$\Downarrow$$

$$\int_{-1}^{1}\frac{d}{dx}\left[\left(1-x^2\right)\left(P_n'P_m - P_m'P\right)\right]dx = \left[m(m+1) - n(n+1)\right]\int_{-1}^{1}P_m(x)P_n(x)dx$$

$$\int_{-1}^{1}\frac{d}{dx}\left[\left(1-x^2\right)\left(P_n'P_m - P_m'P\right)\right]dx = \left(1-x^2\right)\left(P_n'P_m - P_m'P\right)\Big|_{-1}^{1} = 0$$

$$\Downarrow$$

$$\left[m(m+1) - n(n+1)\right]\int_{-1}^{1}P_m(x)P_n(x)dx = 0 \quad \Rightarrow \quad \int_{-1}^{1}P_m(x)P_n(x)dx = 0, \quad \text{since} \quad m \neq n$$

Generating Function for Legendre Polynomials

The generating function for Legendre Polynomials is

$$P(x,r)=\left(1-2xr+r^2\right)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)r^n$$

To see why this is called a generating function, recall the binomial expansion

$$(1-z)^{-1/2}=1+\frac{1}{2}z+\frac{1}{2!}\frac{1}{2}\frac{3}{2}z^2+\frac{1}{3!}\frac{1}{2}\frac{3}{2}\frac{5}{2}z^3+\cdots$$

Let $z=2xr-r^2$, we have

$$P(x,r)=1+\frac{1}{2}\left(2xr-r^2\right)+\frac{3}{8}\left(2xr-r^2\right)^2+\frac{5}{16}\left(2xr-r^2\right)^3+\cdots$$

Re-write it as a power series of ***r***,

$$P(x,r)=1+xr+\left(-\frac{1}{2}+\frac{3}{2}x^2\right)r^2+\left(-\frac{3}{2}x+\frac{5}{2}x^3\right)r^3+\cdots$$

P_0 P_1 P_2 P_3 ……

Theorem 1. (Recurrence Relation of Legendre Polynomial) For each positive integer n and for all $-1 \le x \le 1$,

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

Proof: Differentiating the generating function

$$P(x,r) = \left(1 - 2xr + r^2\right)^{-1/2}$$

w.r.t. r, we obtain

$$\frac{\partial P}{\partial r} = -\frac{1}{2}\left(1 - 2xr + r^2\right)^{-3/2}\left(-2x + 2r\right) = \left(1 - 2xr + r^2\right)^{-3/2}(x - r)$$

⇓

$$\left(1 - 2xr + r^2\right)\frac{\partial P}{\partial r} = \left(1 - 2xr + r^2\right)^{-1/2}\left(x - r\right) = (x - r)P(x,r)$$

Noting that from the property of the generating function, i.e.,

$$P(x,r)=\sum_{n=0}^{\infty}P_n(x)r^n$$

we have

$$\frac{\partial P}{\partial r}=\sum_{n=0}^{\infty}nP_n(x)\,r^{n-1}=\sum_{n=1}^{\infty}nP_n(x)\,r^{n-1}$$

Substitute this into the equation we derived, i.e.,

$$\left(1-2xr+r^2\right)\frac{\partial P}{\partial r}=(x-r)P(x,r)$$

$$\Longrightarrow\quad (1-2xr+r^2)\sum_{n=1}^{\infty}nP_n(x)\,r^{n-1}-(x-r)\sum_{n=0}^{\infty}P_n(x)\,r^n=0$$

$$\sum_{n=1}^{\infty}nP_n(x)\,r^{n-1}-\sum_{n=1}^{\infty}2xnP_n(x)\,r^n+\sum_{n=1}^{\infty}nP_n(x)\,r^{n+1}-\sum_{n=0}^{\infty}xP_n(x)\,r^n+\sum_{n=0}^{\infty}P_n(x)\,r^{n+1}=0$$

$$\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)\,r^n+\sum_{n=1}^{\infty}(-2xn)P_n(x)\,r^n+\sum_{n=2}^{\infty}(n-1)P_{n-1}(x)\,r^n-\sum_{n=0}^{\infty}xP_n(x)\,r^n+\sum_{n=1}^{\infty}P_{n-1}(x)\,r^n=0$$

$$0 = P_1(x) + 2P_2(x)\,r - 2xP_1(x)\,r - xP_0(x) - xP_1(x)\,r + P_0(x)\,r$$

$$+\sum_{n=2}^{\infty}\{\,(n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) - xP_n(x) + P_{n-1}(x)\,\}r^n$$

$$[P_1(x) - xP_0(x)] + [(1+1)P_2(x) - (2+1)xP_1(x) + P_0(x)]\,r$$

$$+\sum_{n=2}^{\infty}\{\,(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)\,\}r^n = 0$$

$$P_1(x) - xP_0(x) = 0$$

$$(1+1)P_2(x) - (2+1)xP_1(x) + P_0(x) = 0$$

and, for $n = 2, 3, 4, \cdots$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

Theorem 2. The coefficient of x^n in $P_n(x)$ is given by $\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}$

Proof: Let c_n be the coefficient of x^n in $P_n(x)$, and consider the recurrence relation

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0$$

The coefficient of x^{n+1} in $(n+1)P_{n+1}(x)$ is equal to $(n+1)c_{n+1}$.

The coefficient of x^{n+1} in $-(2n+1)\,x\,P_n(x)$ is equal to $-(2n+1)\,c_n$.

There is no other x^{n+1} term in the recurrence relation.

Thus coefficient of x^{n+1} is: $(n+1)c_{n+1}-(2n+1)c_n=0 \quad\Rightarrow\quad c_{n+1}=\dfrac{2n+1}{n+1}c_n$

$$c_n=\frac{2n-1}{n}c_{n-1}=\frac{2n-1}{n}\frac{2n-3}{n-1}c_{n-2}=\cdots=\frac{(2n-1)(2n-3)(2n-5)\cdots(1)}{n(n-1)(n-2)\cdots(1)}c_0$$

$$P_0(x)=1 \quad\Rightarrow\quad c_0=1 \quad\Rightarrow\quad c_n=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{n\,!}$$

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Theorem 3. For each positive integer n,

$$nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0$$

Proof. Differentiating the generating function $P(x,r) = \left(1-2xr+r^2\right)^{-1/2}$ w.r.t. x, we obtain

$$\frac{\partial P}{\partial x} = -\frac{1}{2}(-2r)\left(1-2xr+r^2\right)^{-3/2} \Rightarrow \left(1-2xr+r^2\right)\frac{\partial P}{\partial x} = r\left(1-2xr+r^2\right)^{-1/2} = rP(x,r)$$

$$\Downarrow$$

$$\left(1-2xr+r^2\right) = rP(x,r) \Big/ \frac{\partial P}{\partial x}$$

We've proved in Theorem 1 that

$$\left(1-2xr+r^2\right)\frac{\partial P}{\partial r} = (x-r)P(x,r)$$

$$\Downarrow$$

$$\left(1-2xr+r^2\right) = (x-r)P(x,r) \Big/ \frac{\partial P}{\partial r}$$

$$rP(x,r) \Big/ \frac{\partial P}{\partial x} = (x-r)P(x,r) \Big/ \frac{\partial P}{\partial r}$$

$$\boxed{r\frac{\partial P}{\partial r} - (x-r)\frac{\partial P}{\partial x} = 0}$$

Note that $P(x,r)=\sum_{n=0}^{\infty} P_n(x)\, r^n$

$$\Rightarrow \quad \frac{\partial P}{\partial r}=\sum_{n=1}^{\infty} nP_n(x)\, r^{n-1} \quad \& \quad \frac{\partial P}{\partial x}=\sum_{n=0}^{\infty} P_n'(x)\, r^n \quad \& \quad r\frac{\partial P}{\partial r}-(x-r)\frac{\partial P}{\partial x}=0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} nP_n(x)\, r^n-\sum_{n=0}^{\infty} xP_n'(x)\, r^n+\sum_{n=0}^{\infty} P_n'(x)\, r^{n+1}=0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} nP_n(x)\, r^n-\sum_{n=0}^{\infty} xP_n'(x)\, r^n+\sum_{n=1}^{\infty} P_{n-1}'(x)\, r^n=0$$

$$\Downarrow$$

$$-\,xP_0'(x)+\sum_{n=1}^{\infty}\left\{\boxed{nP_n(x)-xP_n'(x)+P_{n-1}'(x)}\right\} r^n \boxed{=0}$$

Prepared by Ben M. Chen

Theorem 4. For each positive integer n,

$$nP_{n-1}(x) - P_n'(x) + xP_{n-1}'(x) = 0$$

Proof: We had in the previous proof the following equalities

$$rP(x,r) = \left(1 - 2xr + r^2\right)\frac{\partial P}{\partial x} \qquad \& \qquad r\frac{\partial P}{\partial r} = (x-r)\frac{\partial P}{\partial x}$$

Next,

$$r\frac{\partial}{\partial r}\left[rP(x,r)\right] = rP(x,r) + r^2\frac{\partial P}{\partial r}$$

$$r\frac{\partial}{\partial r}\left[rP(x,r)\right] = \left(1 - 2xr + r^2\right)\frac{\partial P}{\partial x} + r(x-r)\frac{\partial P}{\partial x}$$

$$= (1 - rx)\frac{\partial P}{\partial x}$$

$$r\frac{\partial}{\partial r}\left[rP(x,r)\right] - (1 - rx)\frac{\partial P}{\partial x} = 0$$

$$P(x,r) = \sum_{n=0}^{\infty} P_n(x)\, r^n$$

$$\Rightarrow \frac{\partial P}{\partial x} = \sum_{n=0}^{\infty} P_n'(x)\, r^n$$

$$r\frac{\partial}{\partial r}\left[rP(x,r)\right] = r\frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} P_n(x)\, r^{n+1}\right]$$

$$= \sum_{n=0}^{\infty} (n+1) P_n(x)\, r^{n+1}$$

$$0 = r\frac{\partial}{\partial r}[rP] - (1 - rx)\frac{\partial P}{\partial x}$$

$$= \sum_{n=0}^{\infty}(n+1)P_n(x)\,r^{n+1} - \sum_{n=0}^{\infty}P_n'(x)\,r^n + \sum_{n=0}^{\infty}xP_n'(x)\,r^{n+1}$$

$$\sum_{n=1}^{\infty}nP_{n-1}(x)\,r^n - \sum_{n=0}^{\infty}P_n'(x)\,r^n + \sum_{n=1}^{\infty}xP_{n-1}'(x)\,r^n = 0$$

$$\sum_{n=0}^{\infty}\left\{\boxed{nP_{n-1}(x) - P_n'(x) + xP_{n-1}'(x)}\right\}r^n - \cancel{P_0'(x)}\ \boxed{= 0}$$

Orthogonal Polynomials

We have shown that if ***m*** and ***n*** are distinct nonnegative integers, then

$$\int_{-1}^{1} P_n(x) P_m(x) dx = 0$$

In view of this, we can say that the Legendre polynomials are orthogonal to each other on the interval [–1, 1]. We also say that the Legendre polynomials form a set of orthogonal polynomials on the interval on [–1, 1].

The orthogonal property can be used to write many functions as series of Legendre polynomials. This will be important in solving certain boundary value problems in partial differential equations.

Let $q(x)$ be a polynomial of degree $\boldsymbol{m}$. We will see how to choose numbers

$\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_m$ such that on $[-1, 1]$

$$q(x) = \sum_{k=0}^{m} \alpha_k P_k(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \cdots + \alpha_j P_j(x) + \cdots + \alpha_m P_m(x)$$

Multiplying the above equation by $\boldsymbol{P_j(x)}$, with $\boldsymbol{0 \le j \le m}$, we obtain

$$P_j(x)q(x) = \alpha_0 P_j(x)P_0(x) + \cdots + \alpha_j P_j(x)P_j(x) + \cdots + \alpha_m P_j(x)P_m(x)$$

Integrating both sides of the above equation from –1 to 1, we have

$$\int_{-1}^{1} P_j(x)q(x)dx = \alpha_0 \underbrace{\int_{-1}^{1} P_j(x)P_0(x)dx}_{0} + \cdots + \alpha_j \int_{-1}^{1} P_j^2(x)dx + \cdots + \alpha_m \underbrace{\int_{-1}^{1} P_j(x)P_m(x)dx}_{0}$$

$$\Rightarrow \int_{-1}^{1} P_j(x)q(x)dx = \alpha_j \int_{-1}^{1} P_j^2(x)dx \Rightarrow \boxed{\alpha_j = \frac{\int_{-1}^{1} P_j(x)q(x)dx}{\int_{-1}^{1} P_j^2(x)dx}, \quad j = 0, 1, \cdots m}$$

Example: Express $q(x) = 1 - 4x^2 + 2x^3$ in terms of $P_0(x)$ to $P_3(x)$,

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x) = \frac{2}{3}\left[\frac{1}{2}\left(3x^2 - 1\right)\right] + \frac{1}{3}$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) = \frac{2}{5}\left[\frac{1}{2}\left(5x^3 - 3x\right)\right] + \frac{3}{5}x$$

$$q(x) = 1 - 4x^2 + 2x^3 = P_0(x) - 4\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] + 2\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right]$$

$$= -\frac{1}{3}P_0(x) + \frac{6}{5}P_1(x) - \frac{8}{3}P_2(x) + \frac{4}{5}P_3(x)$$

$$\int_{-1}^{1} q(x)dx = -\frac{1}{3}\int_{-1}^{1} P_0(x)dx + \frac{6}{5}\overbrace{\int_{-1}^{1} P_1(x)dx}^{0} - \frac{8}{3}\overbrace{\int_{-1}^{1} P_2(x)dx}^{0} + \frac{4}{5}\overbrace{\int_{-1}^{1} P_3(x)dx}^{0} = -\frac{2}{3}$$

Any Polynomial can be written as a Finite Series of Legendre Polynomials.

Theorem 5. Let m and n be nonnegative integers, with $m < n$. Let $q(x)$ be any polynomial of degree m. Then

$$\int_{-1}^{1} q(x) P_n(x)\, dx = 0$$

That is, the integral, from –1 to 1, of a Legendre polynomial multiplied by any polynomial of lower degree, is zero.

Proof: We have shown that for any polynomial $q(x)$ of degree m, there exist scalar α_0, α_1, $\cdots$ α_m such that

$$q(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots + a_m P_m(x)$$

⇓

$$\int_{-1}^{1} q(x) P_n(x)\, dx = a_0 \underbrace{\int_{-1}^{1} P_0(x) P_n(x)\, dx}_{0} + a_1 \underbrace{\int_{-1}^{1} P_1(x) P_n(x)\, dx}_{0} + \cdots + a_m \underbrace{\int_{-1}^{1} P_m(x) P_n(x)\, dx}_{0}$$

$$= 0$$

Theorem 6. For any nonnegative integer n,

$$\int_{-1}^{1}[P_n(x)]^2\,dx = \frac{2}{2n+1}$$

Proof: Let c_n be coefficient of x^n in $P_n(x)$ and also let the coefficient of x^{n-1} in $P_{n-1}(x)$ be c_{n-1}. Define

$$q(x) = P_n(x) - \frac{c_n}{c_{n-1}}xP_{n-1}(x) = c_n x^n + ?\,x^{n-1} + \cdots - \frac{c_n}{c_{n-1}}x\left(c_{n-1}x^{n-1} + ?\,x^{n-2} + \cdots\right)$$

$$= c_n x^n + ?\,x^{n-1} + \cdots - c_n x^n - ?\,x^{n-1} - \cdots = ?\,x^{n-1} + ?\,x^{n-2} + \cdots + ?$$

Thus, $q(x)$ has degree $n-1$ or lower.

$$P_n(x) = \frac{c_n}{c_{n-1}}xP_{n-1}(x) + q(x)$$

$$[P_n(x)]^2 = P_n(x)\left[\frac{c_n}{c_{n-1}}xP_{n-1}(x) + q(x)\right] = \frac{c_n}{c_{n-1}}xP_{n-1}(x)P_n(x) + q(x)P_n(x)$$

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{c_n}{c_{n-1}}\int_{-1}^{1}xP_{n-1}(x)P_n(x)dx + \int_{-1}^{1}q(x)P_n(x)dx = \frac{c_n}{c_{n-1}}\int_{-1}^{1}xP_{n-1}(x)P_n(x)dx$$

(the second integral is marked 0)

Now use the recurrence relation,

$$(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0$$

$$xP_n(x)=\frac{n+1}{2n+1}P_{n+1}(x)+\frac{n}{2n+1}P_{n-1}(x)$$

$$xP_n(x)P_{n-1}(x)=\frac{n+1}{2n+1}P_{n+1}(x)P_{n-1}(x)+\frac{n}{2n+1}[P_{n-1}(x)]^2$$

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{c_n}{c_{n-1}}\int_{-1}^{1}\frac{n+1}{2n+1}P_{n-1}(x)P_{n+1}(x)dx + \frac{c_n}{c_{n-1}}\int_{-1}^{1}\frac{n}{2n+1}[P_{n-1}(x)]^2dx$$

(the first integral is marked 0)

$$= 0 + \frac{c_n}{c_{n-1}}\frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2dx$$

Recall from Theorem 2 that

$$c_n = \frac{1\cdot 3\cdots(2n-1)}{n!}$$

$$\Rightarrow \qquad c_{n-1} = \frac{1\cdot 3\cdots(2n-3)}{(n-1)!} \qquad \Rightarrow \qquad \frac{c_n}{c_{n-1}} = \frac{2n-1}{n}$$

$$\int_{-1}^{1}[P_n(x)]^2 dx = \frac{2n-1}{n}\frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2 dx = \frac{2n-1}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2 dx$$

$$\int_{-1}^{1}[P_n(x)]^2 dx = \frac{2n-1}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2 dx = \frac{(2n-1)(2n-3)}{(2n+1)(2n-1)}\int_{-1}^{1}[P_{n-2}(x)]^2 dx$$

$$= \cdots = \frac{(2n-1)(2n-3)(2n-5)\cdots 3\cdot 1}{(2n+1)(2n-1)(2n-3)\cdots 5\cdot 3}\int_{-1}^{1}[P_0(x)]^2 dx$$

$$= \frac{1}{2n+1}\int_{-1}^{1}[P_0(x)]^2 dx = \frac{2}{2n+1}$$

Finally, we can write any polynomial of degree m as a finite series of Legendre polynomials,

$$q(x) = \sum_{k=0}^{m} a_k P_k(x) \qquad \text{for} \quad -1 \le x \le 1$$

and for $k = 0, 1, 2, \cdots, m$,

$$a_k = \frac{\int_{-1}^{1} q(x) P_k(x) dx}{\int_{-1}^{1} [P_k(x)]^2 dx} = \frac{2k+1}{2} \int_{-1}^{1} q(x) P_k(x) dx$$

Proof. It is a combination of Theorem 6 and the formula derived earlier, i.e.,

$$a_j = \frac{\int_{-1}^{1} P_j(x) q(x) dx}{\int_{-1}^{1} [P_j(x)]^2 dx}, \qquad j = 0, 1, \cdots m$$

Boundary Value Problems in Partial Differential Equations

A partial differential equation is an equation containing one or more partial derivatives, e.g.,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

We seek a solution $u(x, t)$ which depends on the independent variables x and t.

A solution of a partial differential equation is a function which satisfies the equation. For example,

$$u(x,t) = \cos(2x)e^{-4t}$$

is a solution of the above mentioned differential equation since

$$\frac{\partial u}{\partial t} = -4\cos(2x)e^{-4t} \quad = \quad \frac{\partial^2 u}{\partial x^2} = -4\cos(2x)e^{-4t}$$

Order of a Partial Differential Equation

A partial differential equation (p.d.e.) is said to be of order n if it contains an n-th order partial derivative but none of higher order. For example, the following so-called Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is of order 2. The p.d.e. has an order of 5,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^5 u}{\partial t^5} - \frac{\partial u}{\partial t}$$

Linear Case

The general linear **first** order p.d.e. in three variables (with $\boldsymbol{u}$ as a function of the independent variables $\boldsymbol{x}$ and $\boldsymbol{y}$) is

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + f(x, y)u + g(x, y) = 0$$

The general **second** order linear p.d.e. in three variables has the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}+d(x,y)\frac{\partial u}{\partial x}$$
$$+e(x,y)\frac{\partial u}{\partial y}+f(x,y)u+g(x,y)=0$$

Most of the equations we encounter will be in one of these two forms. In both cases, the equation is said to be homogeneous if $g(x,y)=0$ for all (x,y) under consideration and non-homogeneous if $g(x,y)\neq 0$ for some (x,y).

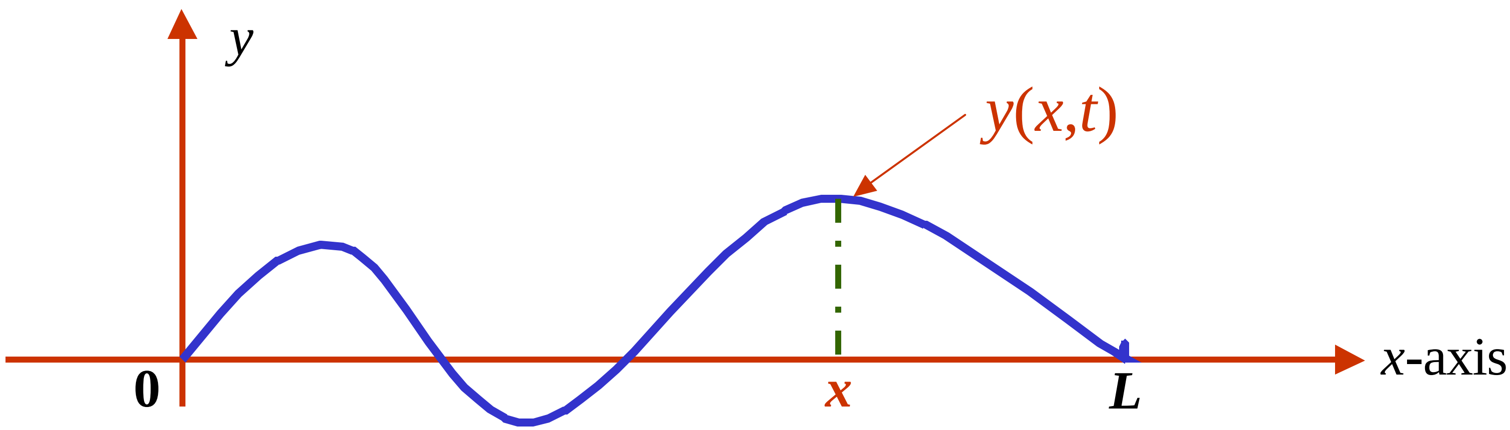
We will focus particularly on equations governing vibration (wave equation) and heat conduction (heat equation) phenomena.

Main Tools: Fourier series, integrals, transforms and Laplace transform.

The Wave Equation

Suppose we have a flexible elastic string stretched between two pegs. We want to describe the ensuing motion if the string is lifted and then released to vibrate in a vertical plane.

Place the x-axis along the length of the string at the rest. At any time t and horizontal coordinate x, let $y(x, t)$ be the vertical displacement of the string.



We want to determine equations which will enable us to solve for $y(x, t)$, thus obtaining a description of the shape of the string at any time.

We will begin by modeling a simplified case. Neglect damping forces such as air resistance and the weight of the string and assume that the tension $T(x, t)$ in the string always acts tangentially to the string. Assume that the string can only move is the vertical direction, i.e., the horizontal tension is constant. Also assume that the mass ρ per unit length is a constant.

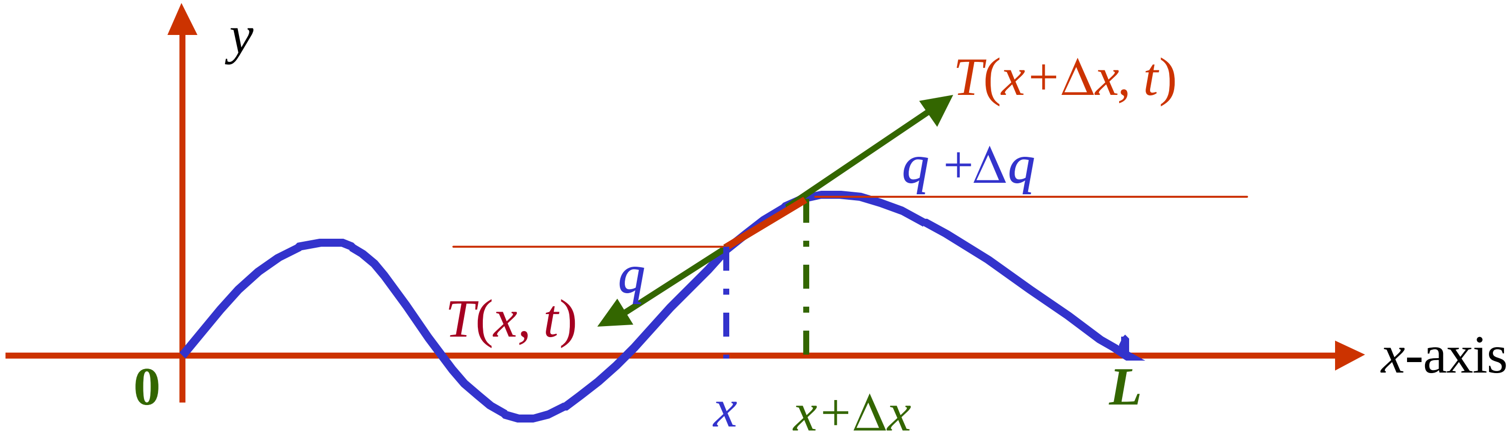
Applying Newton′s 2nd Law of motion to the segment of the string between x and $x + \Delta x$, we have

Net force due to tension = Segment mass

$\times$ Acceleration of the center of mass of the segment

$$\Leftrightarrow \qquad F = ma$$

For small Δx, consideration of the vertical component of the equation gives us approximately:



$$T(x+\Delta x)\sin(q+\Delta q)-T(x,t)\sin q = r\Delta x\frac{\partial^2 y}{\partial t^2}(\bar{x},t), \qquad \bar{x} = x+\frac{1}{2}\Delta x$$

$$\Rightarrow \quad \frac{T(x+\Delta x)\sin(q+\Delta q)-T(x,t)\sin q}{\Delta x} = r\frac{\partial^2 y}{\partial t^2}(\bar{x},t)$$

As a convenience, we write $v(x,t)=T(x,t)\sin q$ i.e., the vertical component of the tension. Hence we have

$$\frac{v(x+\Delta x,t)-v(x,t)}{\Delta x} = r\frac{\partial^2 y}{\partial t^2}(\bar{x},t)$$

$\Delta x \to 0$

$$\frac{\partial v}{\partial x} = r\frac{\partial^2 y}{\partial t^2}$$

Write $h(x, t) = T(x, t)\cos q$, i.e., the horizontal component of the tension at (x, t), then

$$v(x,t) = h(x,t)\tan q = h(x,t)\frac{\partial y}{\partial x}$$

Substituting this into the equation we have just obtained, we have

$$\frac{\partial}{\partial x}\left[h\frac{\partial y}{\partial x}\right] = r\,\frac{\partial^2 y}{\partial t^2}$$

Recall that the horizontal component of the tension of the segment is constant. Hence, h is independent of x and

$$\frac{\partial}{\partial x}\left[h\frac{\partial y}{\partial x}\right] = h\frac{\partial^2 y}{\partial x^2}$$

$$\Downarrow$$

$$h\frac{\partial^2 y}{\partial x^2} = r\,\frac{\partial^2 y}{\partial t^2}$$

$$a^2 = h/r \quad \Rightarrow \quad \boxed{\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}}$$

1-D Wave Equation

The motion of the string will be influenced by both the initial position and the initial velocity of the string. Therefore we must specify initial conditions:

$$y(x,0) = f(x) \qquad \frac{\partial y}{\partial t}(x,0) = g(x) \qquad 0 \le x \le L$$

Next, we consider the boundary conditions. Since the ends of the string are fixed, we have $y(0,t) = y(L,t) = 0,\ \ t \ge 0.$

To be more clear, we can put all of them together, i.e.,

$$\left.\begin{array}{ll} \dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2} & 0 \le x \le L,\ t \ge 0 \\ y(0,t) = y(L,t) = 0 & t \ge 0 \\ y(x,0) = f(x) & 0 \le x \le L \\ \dfrac{\partial y}{\partial t}(x,0) = g(x) & 0 \le x \le L \end{array}\right\}$$

The boundary value problem of 1-D Wave Equation with initial and boundary conditions.

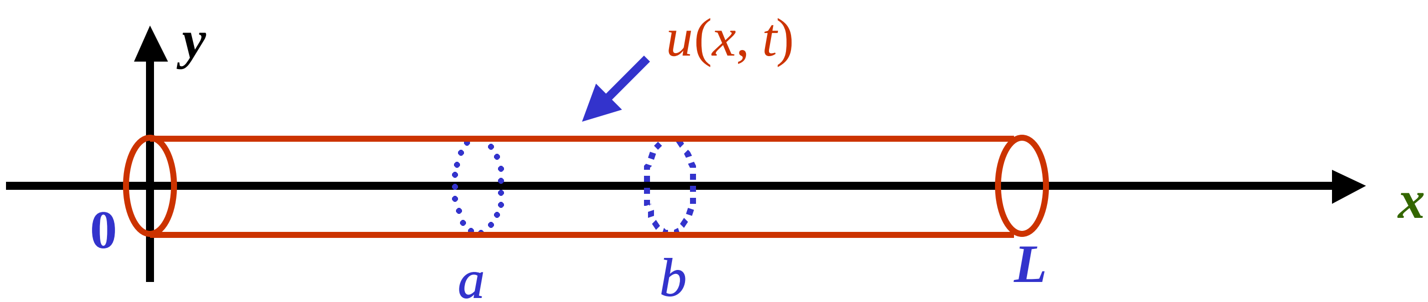
Its solution as expected from the associated physical system must be unique.

The Heat Equation

This is to study temperature distribution in a straight, thin bar under simple circumstances. Suppose we have a straight, thin bar of constant density r and constant cross-sectional area A placed along the x-axis from 0 to L.

Assume that the sides of the bar are insulated and so not allow heat loss and that the temperature on the cross-section of the bar perpendicular to the x-axis at x is a function $u(x, t)$ of x and t, independent of y.

Let the specific heat of the bar be c, and let the thermal conductivity be k, both constant. Now consider a typical segment of the bar between $x = a$ and $x = b$.



By the definition of specific heat, the rate at which heat energy accumulates in this segment of the bar is:

$$\int_a^b c r A \frac{\partial u}{\partial t} dx$$

By Newton′s law of cooling, heat energy flows within this segment from the warmer to the cooler end at a rate equal to ***k*** times the negative of the temperature gradient. Therefore, the net rate at which heat energy enters the segment of bar between a and b at time t is:

$$kA \frac{\partial u}{\partial x}(b,t) - kA \frac{\partial u}{\partial x}(a,t)$$

In the absence of heat production within the segment, the rate at which heat energy accumulates within the segment must balance the rate at which heat energy enters the segment. Hence,

$$\int_a^b c r A \frac{\partial u}{\partial t} dx = kA \frac{\partial u}{\partial x}(b,t) - kA \frac{\partial u}{\partial x}(a,t)$$

Note that

$$kA\frac{\partial u}{\partial x}(b,t) - kA\frac{\partial u}{\partial x}(a,t) = kA\int_a^b \frac{\partial^2 u}{\partial x^2}\,dx$$

$$\Rightarrow \int_a^b \left[crA\frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2} \right] dx = 0 \quad \Rightarrow \quad crA\frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2} = 0$$

This is the so-called heat equation, which is more customary written as:

$$\boxed{\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}}$$

where $a^2 = k/(cr)$ is called the thermal diffusive of the bar. To determine u uniquely, we need boundary conditions (information at the ends of the bar) and initial conditions (temperature throughout the bar at time zero). The p.d.e. together with these pieces of information, constitutes a boundary value problem for the temperature function u.

Problem 1. Both ends of the bar are kept in a constant temperature.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ t > 0)$$

$$u(0,t) = u(L,t) = T \qquad (t > 0)$$

$$u(x,0) = f(x) \qquad (0 < x < L)$$

Problem 2. No heat flows across the ends of the bar.

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ t > 0)$$

$$\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(L,t) = 0 \qquad (t > 0)$$

$$u(x,0) = f(x) \qquad (0 < x < L)$$

Reading assignment: Laplace′s Equation; Poisson′s Equation; Dirichlet and Neumann Problems; Laplace′s Equation in Cylindrical & Spherical Coordinates

Fourier Series Solution of Wave Equation

Recall the wave equation of an initially displaced vibrating string with zero initial velocity,

$\dfrac{\partial^2 y}{\partial t^2} = a^2 \dfrac{\partial^2 y}{\partial x^2}$	$0 \le x \le L$, $t \ge 0$	**Wave equation**
$y(0,t) = y(L,t) = 0$	$t \ge 0$	**Boundary conditions**
$y(x,0) = f(x)$	$0 \le x \le L$	**Initial displacement**
$\dfrac{\partial y}{\partial t}(x,0) = 0$	$0 \le x \le L$	**Initial velocity**

This boundary value problem models the vibration of an elastic string of length L, fastened at the ends, picked up at time zero to assume the shape of the graph of $y(x,0) = f(x)$ and released from rest.

The Fourier method or method of separation of variables is to find a solution of the form

$$y(x,t) = X(x)T(t)$$

with appropriate $X(x)$ *and* $T(t)$ *that solves the above boundary value problem.*

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = X(x)T''(t), \qquad \frac{\partial^2 y}{\partial x^2} = X''(x)T(t)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad \Rightarrow \quad XT'' = a^2 X''T \quad \Rightarrow \quad \frac{X''}{X} = \frac{T''}{a^2 T}$$

We have "separated" x and t; the left hand side is a function of x alone, and the right hand side is a function of t. Since x and t are independent, we can fix the right hand side by choosing $t = t_0$, and the left hand side must be equal to

$$\frac{T''(t_0)}{a^2 T(t_0)}, \text{ which is a constant for all x in } (0, L)$$

$$\Rightarrow \quad \frac{X''}{X} = \frac{T''(t_0)}{a^2 T(t_0)} = -l \ \text{(a separation constant)} = \frac{T''}{a^2 T}$$

$$\Rightarrow \quad X'' + lX = 0 \qquad \& \qquad T'' + l a^2 T = 0$$

These are two ordinary differential equations for X and T.

Next, look at the boundary conditions for $y(x, t)$ and relate them to X and T.

From the condition that the both ends of the string is fixed, we have

$$\left.\begin{aligned} y(0,t) &= X(0)T(t) = 0 \\ y(L,t) &= X(L)T(t) = 0 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} X(0) &= 0 \\ X(L) &= 0 \end{aligned}$$

as we cannot has $T(t) = 0$ if the string is to move. Similarly,

$$\frac{\partial y}{\partial t}(x,0) = 0 \quad \Longrightarrow \quad X(x)T'(0) = 0 \quad \Longrightarrow \quad T'(0) = 0$$

At this point, we have two problems for X and T, namely

$$X'' + \lambda X = 0 \qquad \& \qquad T'' + \lambda a^2 T = 0$$

$$X(0) = X(L) = 0 \qquad\qquad T'(0) = 0$$

A value for λ for which the above problem, either the one associated with X or T, has a nontrivial solution (not identically zero) is called eigenvalue of this problem. The associated nontrivial solution for X or for T is called eigenfunction.

We will consider different cases on λ. We assume that λ is real, as we expect from the physics of the problem.

Case 1: $\lambda = 0$

Then $X'' = 0$, so $\boldsymbol{X(x) = c\,x + d}$ for some constants $\boldsymbol{c}$ and $\boldsymbol{d}$. Then the condition $X(0) = 0$ implies $\boldsymbol{d = 0}$ and $X(L) = c\,L = 0$ implies $\boldsymbol{c = 0}$.

$\Longrightarrow$ $X(x) = 0$ $\Longrightarrow$ $y(x,t) = X(x)T(t) = 0$ if $f(x) \neq 0$.

Case 2: l < 0

For this case, we write $\lambda = -k^2$ with $k > 0$. The equation for X is the given by

$$X'' - k^2 X = 0 \quad \Longrightarrow \quad X(x) = ce^{kx} + de^{-kx}$$

$$\Longrightarrow \quad X(0) = c + d = 0 \quad \Longrightarrow \quad c = -d$$

$$\Longrightarrow \quad X(x) = c\left(e^{kx} - e^{-kx}\right) = 2c\sinh(kx)$$

$$\Longrightarrow \quad X(L) = 2c\sinh(kL) = 0 \quad \Longrightarrow \quad d = c = 0$$

because $\sinh(x) > 0$ if $x > 0$. Thus,

$$y(x,t) = X(x)T(t) = 0 \quad \textit{which is not an admissible solution.}$$

Case 3: $\lambda > 0$

We can write $\lambda = k^2$ with $k > 0$. Then

$$X'' + k^2 X = 0 \quad \Longrightarrow \quad X(x) = c\cos(kx) + d\sin(kx)$$

$$\Longrightarrow \quad X(0) = c = 0 \quad \Longrightarrow \quad X(x) = d\sin(kx)$$

$$\Longrightarrow \quad X(L) = d\sin(kL) = 0$$

We cannot choose $d = 0$ as it will give a trivial solution again. Instead, we have to let

$$\sin(kL) = 0 \quad \Longrightarrow \quad kL = n\pi, \quad n = 1, 2, 3, \cdots \quad \Longrightarrow \quad \lambda = k^2 = \frac{n^2\pi^2}{L^2}$$

Corresponding to each positive integer n, we therefore have a solution for X:

$$X_n(x) = d_n \sin\left(\frac{n\pi}{L}x\right)$$

Now look at the problem for T:

$$l = k^2 = \frac{n^2 p^2}{L^2} \quad \Longrightarrow \quad T'' + \frac{n^2 p^2 a^2}{L} T = 0; \quad T'(0) = 0$$

$$\Longrightarrow \quad T(t) = a \cos\left(\frac{npa}{L} t\right) + b \sin\left(\frac{npa}{L} t\right)$$

$$T'(0) = -\frac{anpa}{L} \sin\left(\frac{npa}{L} t\right) + \frac{bnpa}{L} \cos\left(\frac{npa}{L} t\right)\bigg|_{t=0} = \frac{bnpa}{L} = 0 \quad \Rightarrow \quad b = 0$$

$$\Longrightarrow \quad T_n(t) = a_n \cos\left(\frac{npa}{L} t\right), \quad n = 1, 2, \cdots$$

We now have, for each positive integer n, a function

$$y_n(x,t) = X_n(x) T_n(t) = d_n a_n \sin\left(\frac{np}{L} x\right) \cos\left(\frac{npa}{L} t\right) = A_n \sin\left(\frac{np}{L} x\right) \cos\left(\frac{npa}{L} t\right)$$

which satisfies wave equation and boundary conditions, but not initial position.

In order to satisfy the initial displacement $y(x,0) = f(x)$, we attempt an infinite superposition of the y_n's and write

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{np}{L}x\right)\cos\left(\frac{npa}{L}t\right)$$

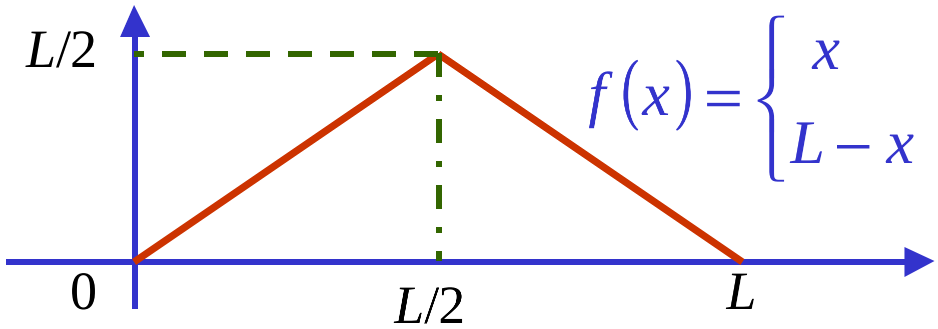
$$\Longrightarrow \quad y(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{np}{L}x\right)$$

Note that this equation is the Fourier sine expansion of $f(x)$ on $[0, L]$. We should choose the A_n's as the Fourier coefficients in this expansion, i.e.,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{np}{L}x\right)dx$$

$$\Longrightarrow \quad y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x)\sin\left(\frac{np}{L}x\right)dx\right]\sin\left(\frac{np}{L}x\right)\cos\left(\frac{npa}{L}t\right)$$

Example: Suppose that initially the string is picked up $L/2$ units at its center point and then release from the rest.



$$f(x) = \begin{cases} x & 0 \le x \le L/2 \\ L - x & L/2 \le x \le L \end{cases}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{np}{L}x\right)dx = \frac{2}{L}\left[\int_0^{L/2} x\sin\left(\frac{np}{L}x\right)dx + \int_{L/x}^{L}(L-x)\sin\left(\frac{np}{L}x\right)dx\right]$$

$$= \frac{2}{L}\left[-\frac{Lx}{np}\cos\left(\frac{npx}{2}\right)\Bigg|_0^{L/2} + \frac{L}{np}\int_0^{L/2}\cos\left(\frac{npx}{L}\right)dx - \frac{L(L-x)}{np}\cos\left(\frac{npx}{L}\right)\Bigg|_{L/2}^{L} - \frac{L}{np}\int_{L/2}^{L}\cos\left(\frac{npx}{L}\right)dx\right]$$

$$= \frac{2}{L}\left[-\frac{L^2}{2np}\cos\left(\frac{np}{2}\right) + \left(\frac{L}{np}\right)^2\sin\left(\frac{npx}{L}\right)\Bigg|_0^{L/2} + \frac{L^2}{2np}\cos\left(\frac{np}{2}\right) - \left(\frac{L}{np}\right)^2\sin\left(\frac{npx}{L}\right)\Bigg|_{L/2}^{L}\right]$$

$$= \frac{2}{L}\left[\left(\frac{L}{np}\right)^2\sin\left(\frac{np}{2}\right) + \left(\frac{L}{np}\right)^2\sin\left(\frac{np}{2}\right)\right] = \boxed{\frac{4L}{n^2p^2}\sin\left(\frac{np}{2}\right) = A_n}$$

$$y(x,t)=\frac{4L}{p^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{np}{2}\right)\sin\left(\frac{npx}{L}\right)\cos\left(\frac{npat}{L}\right)$$

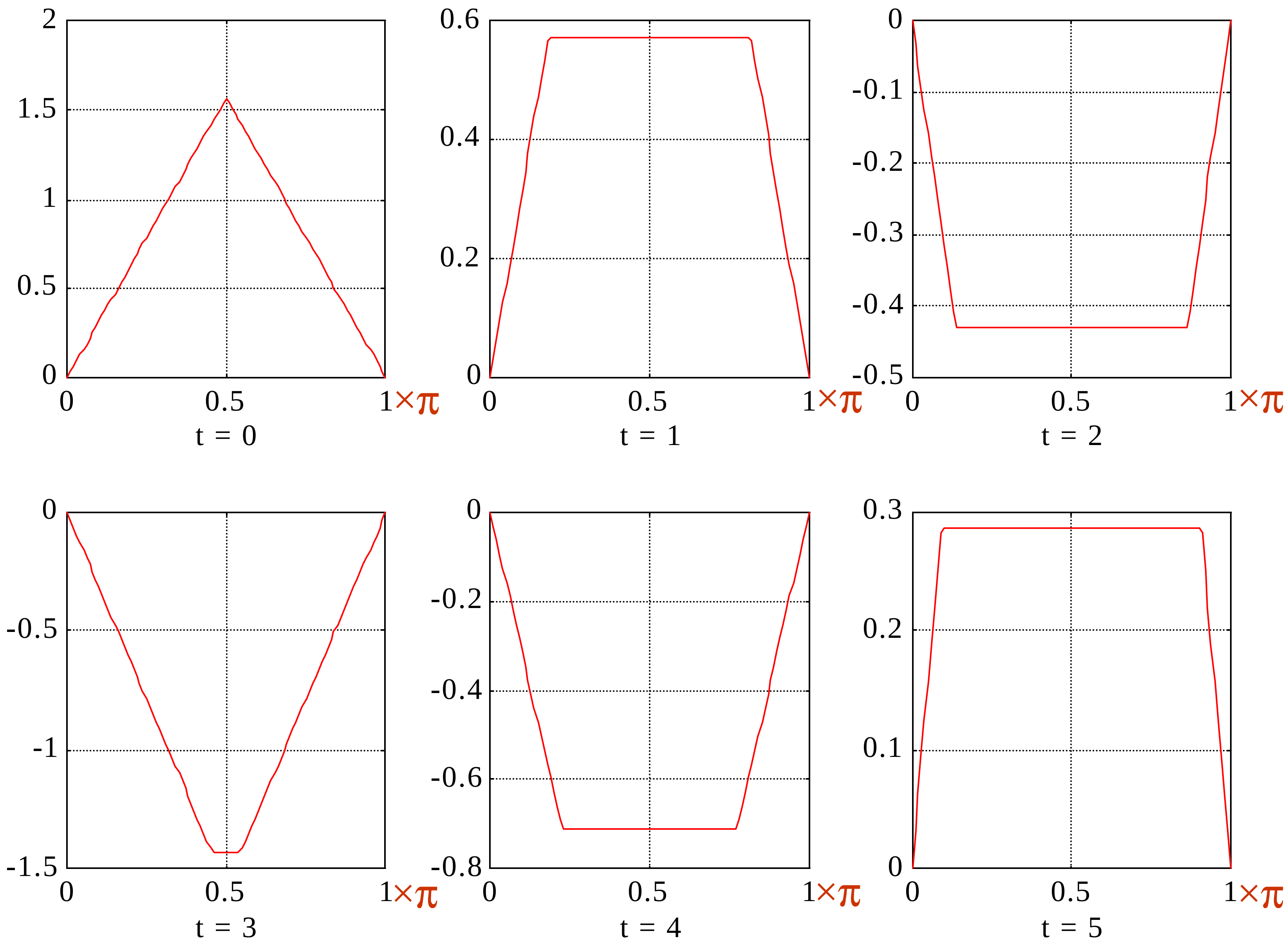
Note that $\sin\left(\frac{np}{2}\right)=\begin{cases}0 & \text{if } n \text{ is even} \\ (-1)^{k+1} & \text{if } n = 2k\text{-}1\end{cases}$

$$A_{2n}=0, \qquad A_{2n-1}=\frac{4L}{(2n-1)^2p^2}(-1)^{n+1}$$

$$y(x,t)=\frac{4L}{p^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)p}{L}x\right]\cos\left[\frac{(2n-1)pa}{L}t\right]$$

The number $\lambda = n^2\pi^2/L^2$ are eigenvalues, and the functions $\sin(n\pi x/L)$, or non-zero multiple thereof, are eigenfunctions. The eigenvalues carry information about the frequencies of the individual sine waves which are superimposed to form the final solution.

Simulation Results: $L = \pi$ and $a = 1$



Prepared by Ben M. Chen

Wave Equation with Zero Initial Displacement but Nonzero Velocity

Now let us consider the case in which the string is released from its horizontal stretched position (zero initial displacement) but with a nonzero initial velocity. The boundary value modeling this phenomenon is

$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$	$0 \le x \le L$, $t \ge 0$	**Wave equation**
$y(0,t) = y(L,t) = 0$	$t \ge 0$	**Boundary conditions**
$y(x,0) = 0$	$0 \le x \le L$	**Initial displacement**
$\frac{\partial y}{\partial t}(x,0) = g(x)$	$0 \le x \le L$	**Initial velocity**

Up to a point the analysis is the same as in the preceding problem. The only difference between this problem and the previous one is that one would have to choose A_n now to meet the initial velocity instead of initial position.

As in the previous case, we set

$$y(x,t) = X(x)T(t)$$

to get

$$X'' + \lambda X = 0 \qquad \& \qquad T'' + \lambda a^2 T = 0$$

$$X(0) = X(L) = 0 \qquad\qquad X(x)T(0) = 0 \Rightarrow T(0) = 0$$

The problem for X is the same as that encountered previously, so the eigen-values are

$$\lambda = \frac{n^2\pi^2}{L^2}$$

For $n = 1, 2, 3, \cdots$ and corresponding eigenfunctions are

$$X_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right)$$

Now, however, we come to the difference between this problem and the preceding one, i.e., the solution for T.

Since we know the values of l, the problem for T can be re-written as

$$T'' + \frac{n^2 p^2 a^2}{L^2} T = 0, \qquad\qquad T(0) = 0$$

The general solution of this differential equation for T is

$$T_n(t) = a_n \cos\left(\frac{npat}{L}\right) + b_n \sin\left(\frac{npat}{L}\right)$$

$$T(0) = 0 \quad \Rightarrow \quad a_n = 0 \quad \Rightarrow \quad T_n(t) = b_n \sin\left(\frac{npat}{L}\right)$$

For each positive integer n, we now have

$$y_n(x,t) = d_n b_n \sin\left(\frac{npx}{L}\right) \sin\left(\frac{npat}{L}\right) = D_n \sin\left(\frac{npx}{L}\right) \sin\left(\frac{npat}{L}\right)$$

Each of these functions satisfies the wave equation, the boundary conditions, and $y_n(x, 0) = 0$, it in general does not satisfy the initial velocity condition.

To satisfy the initial velocity condition, write a superposition

$$y(x,t)=\sum_{n=1}^{\infty} D_n \sin\left(\frac{npx}{L}\right)\sin\left(\frac{npat}{L}\right)$$

$$\Longrightarrow \quad \frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\frac{npa}{L} D_n \sin\left(\frac{npx}{L}\right)\cos\left(\frac{npat}{L}\right)$$

$$\frac{\partial y}{\partial t}(x,0)=g(x) \quad \Longrightarrow \quad g(x)=\sum_{n=1}^{\infty}\frac{npa}{L} D_n \sin\left(\frac{npx}{L}\right)$$

Note that this is a Fourier sine expansion of $g(x)$ on $[0, L]$. By choosing

$$\frac{npa}{L} D_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{npx}{L}\right)dx \quad \Longrightarrow \quad D_n = \frac{2}{npa}\int_0^L g(x)\sin\left(\frac{npx}{L}\right)dx$$

$$\boxed{y(x,t)=\frac{2}{pa}\sum_{n=1}^{\infty}\frac{1}{n}\left[\int_0^L g(x)\sin\left(\frac{npx}{L}\right)dx\right]\sin\left(\frac{npx}{L}\right)\sin\left(\frac{npat}{L}\right)}$$

Example: We consider the same wave equation as in the preceding one but with zero initial displacement and a initial velocity

$$g(x) = \begin{cases} x & 0 \le x \le L/4 \\ 0 & \mathrm{L}/4 < x \le L \end{cases}$$

$$\int_0^L g(x)\sin\left(\frac{npx}{L}\right)dx = \int_0^{L/4} x\sin\left(\frac{npx}{L}\right)dx = \frac{L^2}{n^2p^2}\sin\left(\frac{np}{4}\right) - \frac{L^2}{4np}\cos\left(\frac{np}{4}\right)$$

$$y(x,t) = \frac{2L^2}{p^2a}\sum_{n=1}^{\infty}\left[\frac{1}{n^3p}\sin\left(\frac{np}{4}\right) - \frac{1}{4n^2}\cos\left(\frac{np}{4}\right)\right]\sin\left(\frac{npx}{L}\right)\sin\left(\frac{npat}{L}\right)$$

The Wave Equation with Initial Displacement and Velocity

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ \ t > 0)$$

$$y(0,t) = y(L,t) = 0 \qquad (t > 0)$$

$$y(x,0) = f(x) \qquad (0 < x < L)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \qquad (0 < x < L)$$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad 0 \le x \le L,\ \ t \ge 0$$

$$y(0,t) = y(L,t) = 0 \qquad t \ge 0$$

$$y(x,0) = f(x) \qquad 0 \le x \le L$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \qquad 0 \le x \le L$$

$+$

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad 0 \le x \le L,\ \ t \ge 0$$

$$y(0,t) = y(L,t) = 0 \qquad t \ge 0$$

$$y(x,0) = 0 \qquad 0 \le x \le L$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \qquad 0 \le x \le L$$

Example: Consider the wave equation with usual boundary and with initial displacement,

$$f(x)=\begin{cases} x & 0 \le x \le L/2 \\ L-x & L/2 \le x \le L \end{cases}$$

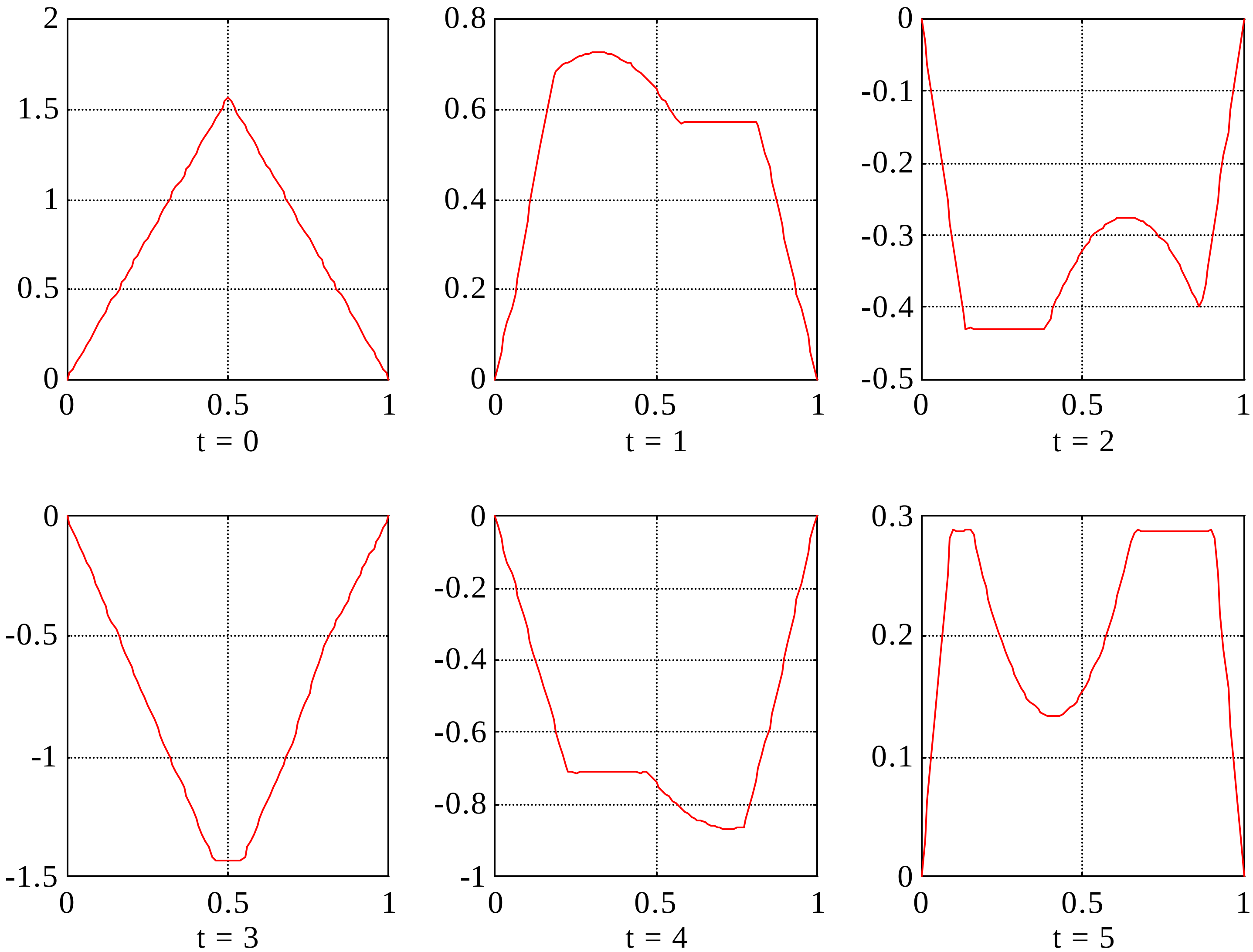
and initial velocity

$$g(x)=\begin{cases} x & 0 \le x \le L/4 \\ 0 & L/4 < x \le L \end{cases}$$

Then it follows from the previous examples, the solution to the above problem is given by

$$y(x,t)=\frac{4L}{p^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{np}{2}\right)\sin\left(\frac{npx}{L}\right)\cos\left(\frac{npat}{L}\right)$$
$$+\frac{2L^2}{p^2 a}\sum_{n=1}^{\infty}\left[\frac{1}{n^3 p}\sin\left(\frac{np}{4}\right)-\frac{1}{4n^2}\cos\left(\frac{np}{4}\right)\right]\sin\left(\frac{npx}{L}\right)\sin\left(\frac{npat}{L}\right)$$

Simulation Results: L = p and a = 1



Fourier Series Solution of the Heat Equation

The Ends of the Bar Kept at Zero Temperature: We want to determine the temperature distribution $u(x,t)$ in a thin homogeneous bar of length L, given the initial temperature distribution throughout the bar at time $t = 0$, if the ends are maintained at zero temperature for all time.

The boundary value problem is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L,\ \ t > 0) \qquad \textbf{Heat Equation}$$

$$u(0,t) = u(L,t) = 0 \qquad (t > 0) \qquad \textbf{Boundary Conditions}$$

$$u(x,0) = f(x) \qquad (0 < x < L) \qquad \textbf{Initial Temperature}$$

The procedure that we are going to use to solve the above problem is precisely the same as the one in solving the wave equation. All the dirty works were done already. The rest is pretty simple.

We will apply the separation of variables method and seek a solution

$$u(x,t) = X(x)T(t)$$

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad X(x)T'(t) = a^2 X''(x)T(t) \quad \Rightarrow \quad \frac{T'(t)}{T(t)} = a^2 \frac{X''(x)}{X(x)}$$

Since x and t are independent variables, both sides of this equation must be equal to the same constant.

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -l \quad \Rightarrow \quad X'' + lX = 0 \quad \text{and} \quad T' + l a^2 T = 0$$

$$u(0,t) = 0 \quad \Rightarrow \quad X(0)T(t) = 0 \quad \Rightarrow \quad X(0) = 0$$

$$u(L,t) = 0 \quad \Rightarrow \quad X(L)T(t) = 0 \quad \Rightarrow \quad X(L) = 0$$

$$\Rightarrow \quad X'' + lX = 0, \quad X(0) = X(L) = 0 \quad \& \quad T' + l a^2 T = 0$$

Unlike the wave equation, this equation for T is of first order, with no boundary condition.

The boundary value problem for X, however, is identical with that encountered with the wave equation, so we have eigenvalues and eigenfunctions

$$\ddot{e}_n = \frac{n^2 p^2}{L^2} \qquad \& \qquad X_n(x) = d_n \sin\left(\frac{npx}{L}\right)$$

With the values we have for l, the differential equation for T is

$$T' + \frac{a^2 n^2 p^2}{L^2} T = 0 \quad \Rightarrow \quad T_n(t) = a_n e^{-n^2 p^2 a^2 t / L^2}$$

$$\Rightarrow \quad u_n(x,t) = X_n(x) T_n(t) = A_n \sin\left(\frac{npx}{L}\right) e^{-n^2 p^2 a^2 t / L^2}$$

Each $u_n(x,t)$ satisfies the heat equation and both boundary conditions, but in general, none of them will satisfy the initial condition, i.e.,

$$u(x,0) = f(x) \qquad (0 < x < L)$$

Thus, we should attempt an infinite superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{npx}{L}\right) e^{-n^2 p^2 a^2 t / L^2}$$

$$\Longrightarrow \qquad u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{npx}{L}\right)$$

This is the Fourier series expansion of f on $[0, L]$. Hence, choose the A_n's as the Fourier coefficients:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{npx}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{npx}{L}\right)dx$$

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x)\sin\left(\frac{npx}{L}\right)dx\right]\sin\left(\frac{npx}{L}\right)e^{-n^2 p^2 a^2 t / L^2}$$