EE 2462 Engineering Math III, Part 1

Power Series, Special Functions & Boundary Value Problems

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Course Outline:

Power Series: Sequences and Series; Convergence and Divergence; A Test for Divergence; Comparison Tests for Positive Series; The Ratio Test for Positive Series; Absolute Convergence; Power Series.

Special Functions: Bessel's Equation and Bessel's Functions; The Gamma Function; Solution of Bessel's Equation in Terms of the Gamma Function; Modified Bessel's Equations; Applications of Bessel's Functions; Legendre's Equation and Legendre Polynomials.

Partial Differential Equations: Boundary Value Problems in Partial Differential Equations; Wave Equation; Heat Equation; Laplace's Equation; Poisson's Equation; Dirichlet and Neumann Problems.

Textbooks:

1. P. V. O'Neil, *Advanced Engineering Mathematics*, PWS-Kent Publishing Company, 4th Edition, 1995.

2. E. Kreyszig, *Advanced Engineering Mathematics*, John Wiley, 7th Edition, 1995.

Lecture Notes and Tutorial Sets:

PDF files of the lecture notes & tutorial sets of this part can be downloaded from my home page at http://www.ee.nus.edu.sg/~bmchen. **Tutorial starts on the fourth week!**

1. Sequences and Series

If a set of numbers is so arranged that there is a first, a second, a third, and so on, it constitutes a *sequence*. We indicate a sequence by the general term in braces, $\{f(n)\}$. As examples of sequences we have

$$\{u_n\} : u_1, u_2, \cdots, u_n, \cdots; \tag{1.1}$$

$$\left\{\frac{1}{2^n}\right\} : \frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2^n}, \cdots; \tag{1.2}$$

$$\left\{\frac{1}{n}\right\} : 1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots. \tag{1.3}$$

With any sequence $\{u_n\}$ we can associate an array

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots, \tag{1.4}$$

called a *series*. If all but a finite number of terms, say, those with $n > N$, are equal to zero, then the array takes the form

$$\sum_{i=1}^{N} u_i = u_1 + u_2 + u_3 + \cdots + u_N. \tag{1.5}$$

This has a sum in the ordinary sense. And this sum is called the *value* of the finite series.

1.1. Convergence and Divergence

The n-th *partial sum* of the infinite series

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots, \tag{1.6}$$

s_n, is defined as the sum of its first n terms. That is

$$s_n = \sum_{i=1}^{n} u_i = u_1 + u_2 + u_3 + \cdots + u_n. \tag{1.7}$$

The partial sum form a new sequence $\{s_n\}$. If, as n increases and tends to infinity the sequence of numbers s_n approaches a finite limit L, the series

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots, \tag{1.8}$$

converges. And we write

$$\lim_{n\to\infty} s_n = L. \tag{1.9}$$

We say that the infinite series converges to L and that L is the *value*, or *sum*, of the series.

If the sequence does not approach a limit, then the series is *divergent* and we do not assign any value to it.

1.2. A Test for Divergence

Suppose that the series

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots, \tag{1.10}$$

converges. Then, since $(n-1) \to \infty$ when $n \to \infty$, we have

$$\lim_{n\to\infty} s_n = L \quad \text{and} \quad \lim_{n\to\infty} s_{n-1} = L. \tag{1.11}$$

But $s_n = s_{n-1} + u_n$, or $u_n = s_n - s_{n-1}$. Hence, as $n \to \infty$,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0. \tag{1.12}$$

Hence, as $n \to \infty$, $u_n \to 0$ in any convergent series. Thus, we have established the following result:

If u_n does not tend to zero as n becomes infinite, the series

$$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots, \tag{1.13}$$

is divergent.

Example 1. Show that the series with

$$u_n = \frac{n}{2n+1}, \tag{1.14}$$

that is, the series

$$\frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \cdots + \frac{n}{2n+1} + \cdots, \tag{1.15}$$

diverges.

SOLUTION. Here

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2+1/n} = \frac{1}{2} \neq 0. \tag{1.16}$$

Hence, it diverges.

Example 2. Show that the series with odd terms $\frac{n+1}{n}$ and even terms $1/n$, namely, the series

$$2 + \frac{1}{2} + \frac{4}{3} + \frac{1}{4} + \cdots + \frac{m}{m-1} + \frac{1}{m} + \frac{m+2}{m+1} + \cdots, \tag{1.17}$$

diverges.

SOLUTION. Here, for large n, there are (odd) terms, $1 + 1/n$, near 1, and also (even) terms, $1/n$, near 0. Thus, no limit is approached by u_n; the terms near 1 show that we cannot have $u_n \to 0$.

1.3. Comparison Tests for Positive Series

A *positive series* is one with each $u_n \geq 0$. We have the following results:

1. *Let* $\sum v_n$ *be a positive series known to converge. If* $0 \leq u_n \leq v_n$ *for all* n*, then the series* $\sum u_n$ *converges.*

2. *Let* $\sum V_n$ *be a positive series known to diverge. If* $u_n \geq V_n$ *for all* n*, then the series* $\sum u_n$ *diverges.*

1.4. The Ratio Test for Positive Series

For a positive series $\sum u_n$, let us define the test ratio

$$t_n = \frac{u_{n+1}}{u_n}. \tag{1.18}$$

Suppose that, as $n \to \infty$, $t_n \to T$. Then the ratio test asserts the following:

For a positive series $\sum u_n$*, with*

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = T, \tag{1.19}$$

the series converges if $T < 1$*. And the series diverges if* $T > 1$*. For the value* $T = 1$*, no conclusion can be drawn.*

Example 1. Test the series with

$$u_n = \frac{(n-1)!}{n^{n-1}} \tag{1.20}$$

for convergence. (note that $u_1 = 1$, since $0! = 1$.)

SOLUTION. The test ratio is

$$t_n = \frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)^n} \frac{n^{n-1}}{(n-1)!} \tag{1.21}$$

Since $n! = n(n-1)!$, we have

$$t_n = \left(\frac{n}{n+1}\right)^n = \frac{1}{(1+1/n)^n} \to \frac{1}{e} \approx \frac{1}{2.7} < 1. \tag{1.22}$$

Hence, the series converges.

1.5. Absolute Convergence

Consider a series with general term u_n, which may be positive, zero, or negative. Then, with $\sum u_n$, we associate the positive series $\sum |u_n|$, obtained by taking absolute values. If this latter series converges, the first series is said to *converge absolutely*. We have the following result:

If $\sum |u_n|$ *converges, then* $\sum u_n$ *converges.*

Since the series of absolute values is a positive series, its convergence can be tested by the comparison or ratio test.

If the series of absolute values diverges, the original series may still converge, and in this case it is said to *converge conditionally.*

1.6. Power Series

Any infinite series of the form

$$A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots \tag{1.23}$$

is called a *power series.* If we call the first term u_0, we may write it

$$\sum_{n=0}^{\infty} u_n(x) \qquad \text{where } u_n(x) = A_n(x-a)^n. \tag{1.24}$$

Let us suppose that the absolute value of the ratio of successive coefficients,

$$\lim_{n\to\infty} \left| \frac{A_{n+1}}{A_n} \right| = L \neq 0. \tag{1.25}$$

Then, the power series

$$\sum_{n=0}^{\infty} u_n(x) \qquad \text{where } u_n(x) = A_n(x-a)^n, \tag{1.26}$$

converges absolutely for any x such that

$$|x-a| < \frac{1}{L}. \tag{1.27}$$

PROOF. This makes the limit of the ratio of the numerical values of successive terms in the series

$$\sum_{n=0}^{\infty} u_n(x) \qquad \text{where } u_n(x) = A_n(x-a)^n, \tag{1.28}$$

i.e.,

$$\lim_{n\to\infty} \left| \frac{A_{n+1}(x-a)^{n+1}}{A_n(x-a)^n} \right| = L|x-a| < 1. \tag{1.29}$$

Hence, the positive series with general term $|A_n(x-a)^n|$ converges by the ratio test, and the series converges absolutely.

We can also show that the series

$$\sum_{n=0}^{\infty} u_n(x) \qquad \text{where } u_n(x) = A_n(x-a)^n \tag{1.30}$$

diverges for any x_1 such that

$$|x_1 - a| > \frac{1}{L}. \tag{1.31}$$

Example. Find the open interval of absolute convergence of the power series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots. \tag{1.32}$$

SOLUTION. Here

$$\left|\frac{x^{n+1}/(n+1)}{x^n/n}\right| = \frac{n|x|}{n+1} = \frac{|x|}{1+(1/n)} \to |x|. \tag{1.33}$$

Thus, the interval of absolute convergence is $|x| < 1$, or $-1 < x < 1$.

1.7. Operations with Power Series

For any two convergent series

$$\sum_{n=0}^{\infty} u_n = U \quad \text{and} \quad \sum_{n=0}^{\infty} v_n = V, \tag{1.34}$$

the series may be added term by term and

$$\sum_{n=0}^{\infty} (u_n + v_n) = U + V. \tag{1.35}$$

It is not necessarily true that the product series, with general term w_n, where

$$w_n = u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \cdots + u_n v_0, \tag{1.36}$$

converges, or if it does converge that its sum is UV. But this is always true if both series converge absolutely.

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2. Bessel's Equation and Bessel Functions of the 1st Kind

The differential equation

$$x^2y'' + xy' + (x^2 - v^2)y = 0 \tag{2.1}$$

is called Bessel's equation of order v with $v \geq 0$. The following is the so-called Frobenius Solution (or Method of Frobenius) to the above Bessel's equation: Define a Frobenius Series

$$y(x) = \sum_{n=0}^{\infty} c_n x^{n+r} \tag{2.2}$$

where c_n and r are unknown constants to be determined. We have

$$y'(x) = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \tag{2.3}$$

$$y''(x) = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} \tag{2.4}$$

Substituting Equations (2.2), (2.3) and (2.4) into the Bessel's Equation in (2.1), we have

$$\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} c_n v^2 x^{n+r} = 0$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} c_n(n+r)^2 x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} - \sum_{n=0}^{\infty} c_n v^2 x^{n+r} = 0$$

$$\Downarrow$$

$$c_0 r^2 x^r + c_1(r+1)^2 x^{r+1} - c_0 v^2 x^r - c_1 v^2 x^{r+1} + \sum_{n=2}^{\infty} \left[c_n(n+r)^2 + c_{n-2} - c_n v^2 \right] x^{n+r} = 0$$

$$\Downarrow$$

$$(r^2 - v^2)c_0 x^r + [(r+1)^2 - v^2]c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[c_n(n+r)^2 + c_{n-2} - c_n v^2 \right] x^{n+r} = 0$$

Let the coefficient of each power of x be zero. Also, assume that $c_0 \neq 0$, we find from the above derivation that the coefficient of x^r is:

$$F(r) = r^2 - v^2 = 0$$

This equation has two solutions, $r = v$ and $r = -v$.

Case 1: First, let us take the solution $r = v$. Substituting this solution into the coefficient of x^{r+1}, we get

$$(2v+1)c_1 = 0$$

Since $v \geq 0$, this equation implies that

$$c_1 = 0$$

From the coefficient of x^{n+r}, we obtain the recurrence relation:

$$n(n+2v)c_n + c_{n-2} = 0$$

for $n = 2, 3, 4, \cdots$

$$\Downarrow$$

$$c_n = -\frac{1}{n(n+2v)}c_{n-2}, \quad n = 2, 3, 4, \cdots$$

Since $c_1 = 0$, the recurrence relation above gives

$$c_3 = 0, \;\; c_5 = 0, \;\; \cdots$$

In general $c_n = 0$ with n being **odd** positive integers.

For even number of $n = 2m$, we have

$$\begin{aligned} c_{2m} &= -\frac{1}{2m(2m+2v)}c_{2m-2} \\ &= -\frac{1}{2^2 m(m+v)}c_{2m-2} \\ &= -\frac{1}{2^2 m(m+v)}\frac{-1}{2^2(m-1)(m+v-1)}c_{2m-4} \\ &= \frac{c_{2m-4}}{2^4 m(m-1)(m+v)(m+v-1)} \\ &= \cdots\cdots \\ &= \frac{(-1)^m c_0}{2^{2m}\cdot[m(m-1)\cdots 1]\cdot[(m+v)(m+v-1)\cdots(v+1)]} \end{aligned}$$

This gives us one solution of Bessel's equation of order v, i.e.,

$$y_1(x) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+v}}{2^{2m} m!(m+v)(m+v-1)\cdots(v+1)}$$

$$\Downarrow$$

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n} n!(n+v)(n+v-1)\cdots(v+1)}$$

We will re-written the above solution in terms of the Gamma function. But, first of all let us introduce this Gamma function and examine its properties.

The Gamma Function

For $x > 0$, define a so-called Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

If $x > 0$, then $\Gamma(x+1) = x\,\Gamma(x)$. □

Proof:

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$$

$$= -t^x e^{-t}\Big|_0^\infty + \int_0^\infty x t^{x-1} e^{-t} dt$$

$$= x \int_0^\infty t^{x-1} e^{-t} dt$$

$$= x\,\Gamma(x)$$

Q.E.D.

For any positive integer n,

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\,\Gamma(n-1)$$

$$= \cdots\cdots$$

$$= n(n-1)(n-1)\cdots(n-n+1)\,\Gamma(1)$$

$$= n!\,\Gamma(1)$$

Observing that

$$\Gamma(1) = \int_0^\infty e^{-t} dt = -e^{-t}\Big|_0^\infty = 1$$

Hence

$$\Gamma(n+1) = n!$$

If $v \geq 0$, but v is not necessarily an integer, a similar property holds:

$$\begin{aligned}
\Gamma(n+v+1) &= (n+v)\ \Gamma(n+v) \\
&= (n+v)(n+v-1)\ \Gamma(n+v-1) \\
&= \cdots\cdots \\
&= (n+v)(n+v-1)\cdots(n+v-n+1)\ \Gamma(v+1) \\
&= (n+v)(n+v-1)\cdots(v+1)\ \Gamma(v+1)
\end{aligned}$$

which implies

$$(n+v)(n+v-1)\cdots(v+1) = \frac{\Gamma(n+v+1)}{\Gamma(v+1)}$$

This is known as the factorial property of the Gamma function.

Although the improper integral defining $\Gamma(x)$ converges only if $x > 0$, it is possible to define $\Gamma(x)$ if x is negative (but not an integer). We can write

$$\Gamma(x) = \frac{1}{x}\ \Gamma(x+1)$$

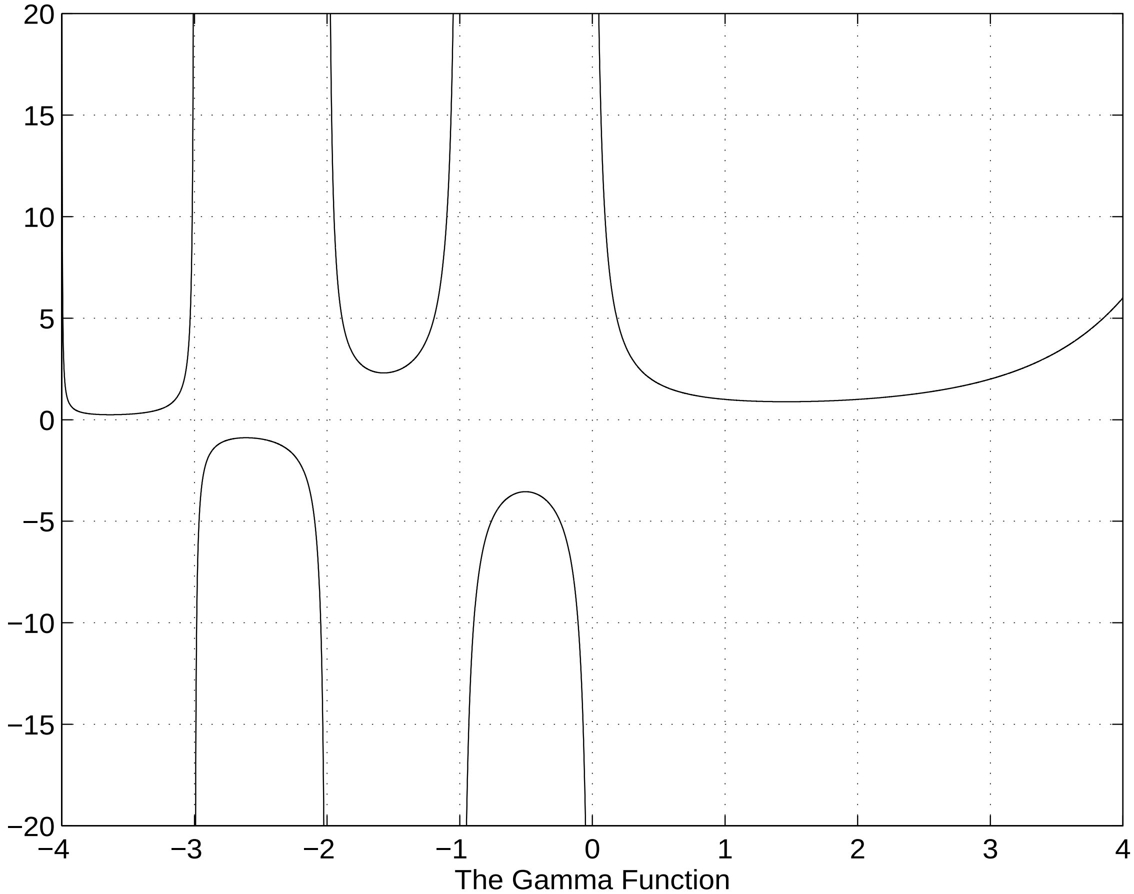
This holds for all $x > 0$. Now, if $-1 < x < 0$, then $0 < x+1 < 1$ and $\Gamma(x+1)$ is properly defined. We can therefore define $\Gamma(x)$ for $-1 < x < 0$, e.g.,

$$\Gamma\left(-\frac{1}{2}\right) = \frac{1}{-\frac{1}{2}}\ \Gamma\left(-\frac{1}{2}+1\right) = -2\ \Gamma\left(\frac{1}{2}\right)$$

Having defined $\Gamma(x)$ on $(-1, 0)$, suppose now that $-2 < x < -1$, then $-1 < x+1 < 0$, we can again follow the same procedure to define $\Gamma(x)$ for $-2 < x < -1$, e.g.,

$$\Gamma\left(-\frac{3}{2}\right) = \frac{1}{-\frac{3}{2}}\ \Gamma\left(-\frac{3}{2}+1\right) = -\frac{2}{3}\ \Gamma\left(-\frac{1}{2}\right) = +\frac{4}{3}\ \Gamma\left(\frac{1}{2}\right)$$

Clearly, we can continue on this process forever by moving to the left over the real line and defining $\Gamma(x)$ on every interval $(k-1, k)$ once it has been defined on the interval $(k, k+1)$ for any negative integer k.



The Gamma Function Over Interval $(-4, 4)$.

Solution of Bessel's Equation in Terms of the Gamma Function

Recall that the first solution we have obtained for the Bessel's equation, i.e.,

$$y_1(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n} \cdot n! \cdot (n+v)(n+v-1)\cdots(v+1)}$$

$$= c_0 \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(v+1) x^{2n+v}}{2^{2n} \cdot n! \cdot \Gamma(n+v+1)}$$

The second expression is true because

$$(n+v)(n+v-1)\cdots(v+1) = \frac{\Gamma(n+v+1)}{\Gamma(v+1)}$$

If we choose

$$c_0 = \frac{1}{2^v \Gamma(v+1)}$$

we obtain

$$y_1(x) = J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} \cdot n! \cdot \Gamma(n+v+1)}$$

$J_v(x)$ is called a *Bessel Function* of the 1st kind of order v. This series converges for all x (show it).

Exercise Problems: Show that

1. $J'_v(x) = \frac{1}{2}[J_{v-1}(x) - J_{v+1}(x)]$
2. $vJ_v(x) = \frac{x}{2}[J_{v-1}(x) + J_{v+1}(x)]$
3. $\frac{d}{dt}[t^v J_v(t)] = t^v J_{v-1}(t)$

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Solutions to the Exercise Problems

1) To verify

$$J'_v(x) = \frac{1}{2}[J_{v-1}(x) - J_{v+1}(x)]$$

Note that

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!\ \Gamma(n+v+1)}$$

$$\Downarrow$$

$$J_v'(x) = \sum_{n=0}^{\infty} (2n+v) \frac{(-1)^n x^{2n+v-1}}{2^{2n+v} n!\ \Gamma(n+v+1)}$$

$$\Downarrow$$

$$J_{v-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v-1}}{2^{2n+v-1} n!\ \Gamma(n+v)} = \sum_{n=0}^{\infty} \frac{2(n+v)(-1)^n x^{2n+v-1}}{2^{2n+v} n!\ \Gamma(n+v+1)}$$

The second equality holds because $\Gamma(n+v+1) = (n+v)\Gamma(n+v)$. Therefore, we have

$$J_v'(x) - \frac{1}{2} J_{v-1}(x) = \sum_{n=0}^{\infty} \frac{n(-1)^n x^{2n+v-1}}{2^{2n+v} n!\ \Gamma(n+v+1)}$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+v-1}}{2^{2n+v} (n-1)!\ \Gamma(n+v+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+v+1}}{2^{2n+v+2} n!\ \Gamma(n+v+2)}$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v+1}}{2^{2n+v+1} n!\ \Gamma(n+v+2)}$$

$$= -\frac{1}{2} J_{v+1}(x)$$

That is

$$J_v'(x) = \frac{1}{2}[J_{v-1}(x) - J_{v+1}(x)]$$

Q.E.D.

2) To verify

$$v J_v(x) = \frac{x}{2}[J_{v-1}(x) + J_{v+1}(x)]$$

First note that

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!\ \Gamma(n+v+1)}$$

and

$$\frac{x}{2} J_{v-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!\ \Gamma(n+v)}$$

Then we have

$$v J_v(x) - \frac{x}{2} J_{v-1}(x) = \sum_{n=0}^{\infty} \frac{[v-(n+v)](-1)^n x^{2n+v}}{2^{2n+v} n!\ \Gamma(n+v+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-n)(-1)^n x^{2n+v}}{2^{2n+v} n!\ \Gamma(n+v+1)}$$

$$= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+v}}{2^{2n+v}(n-1)!\,\Gamma(n+v+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2n+2+v}}{2^{2n+2+v} n!\,\Gamma(n+v+2)}$$

$$= \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{2n+v+1}}{2^{2n+v+1} n!\,\Gamma(n+v+2)}$$

$$= \frac{x}{2} J_{v+1}(x)$$

That is,

$$v J_v(x) = \frac{x}{2}[J_{v-1}(x) + J_{v+1}(x)]$$

Q.E.D.

3) To verify

$$\frac{d}{dt}[t^v J_v(t)] = t^v J_{v-1}(t)$$

We let

$$y = t^v J_v(t)$$

$$= t^v \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+v}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2v}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

Then we have

$$\frac{dy}{dt} = \sum_{n=0}^{\infty} \frac{(2n+2v)(-1)^n t^{2n+2v-1}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+2v-1}}{2^{2n+v-1} n!\,\Gamma(n+v)}$$

$$= t^v \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+v-1}}{2^{2n+v-1} n!\,\Gamma(n+v)}$$

$$= t^v J_{v-1}(t)$$

Again, we have used the factorial property of the Gamma function in the above derivation. That is

$$\Gamma(n+v+1) = (n+v)\,\Gamma(n+v)$$

Q.E.D.

We have considered the case in which v is any nonnegative number. We will now consider the problem of finding a second, linearly independent solution of Bessel's equation. Recall that

$$F(r) = r^2 - v^2 = 0$$

has two roots $r_1 = v$ and $r_2 = -v$.
Case 1: v is not an integer.

Theorem: If v is not an integer, then two linearly independent solutions of Bessel's equation of order v are J_v and J_{-v}, where

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+v}}{2^{2n+v} n!\,\Gamma(n+v+1)}$$

Thus, the general solution of Bessel's equation of order v, where v is not an integer, is

$$y(x) = \alpha_1 J_v(x) + \alpha_2 J_{-v}(x)$$

where α_1 and α_2 are real scalars.

Case 2: v is an integer.

If v is an integer, say $v = k$, then $J_v(x)$ and $J_{-v}(x)$ are solutions of Bessel's equation of order v, but they are NOT linearly independent. This fact can be verified from the following arguments: First note that

$$J_{-v}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-v}}{2^{2n-v} n!\,\Gamma(n-v+1)}$$

Observing the values of Gamma function at $0, -1, -2, \cdots\cdots$, they go to infinity. Thus we have

$$\Gamma(n-v+1) \to \infty \quad \text{or} \quad \frac{1}{\Gamma(n-v+1)} \to 0$$

as $v \to k$ for $n = 0, 1, 2, \cdots, k-1$. Hence

$$J_{-k}(x) = \sum_{n=k}^{\infty} \frac{(-1)^n x^{2n-k}}{2^{2n-k} n!\,\Gamma(n-k+1)}$$

In this, let the variable of summation be changed from n to m by the substitution $n = m + k$. Then

$$J_{-k}(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+k} x^{2(m+k)-k}}{2^{2(m+k)-k}(m+k)!\ \Gamma(m+k-k+1)}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (-1)^k x^{2m+k}}{2^{2m+k}(m+k)!\ \Gamma(m+1)}$$

$$= (-1)^k \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+k}}{2^{2m+k}(m+k)!\ \Gamma(m+1)} = (-1)^k \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+k}}{2^{2m+k}(m+k)!m!}$$

$$= (-1)^k \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+k}}{2^{2m+k}\ \Gamma(m+k+1)m!} = (-1)^k J_k(x)$$

Thus, we need to search for another linearly independent solution of Bessel's equation. The is the subject of the next topic.

3. Bessel Function of the 2nd Kind

For the Bessel's equation of order v:

$$x^2 y'' + xy' + (x^2 - v^2)y = 0$$

in which $v \geq 0$, the general solution is

$$y(x) = \alpha_1 J_v(x) + \alpha_2 J_{-v}(x)$$

if v is not an integer. Now, in the case in which v is a nonnegative integer, say $v = k$, for some nonnegative k, we have one solution $J_k(x)$ of Bessel's equation but have not yet derived a second, linearly independent solution. In what follows, we will try to find this second solution.

A Second Solution of Bessel's Equation for the Case $v = k = 0$

Let us try a solution of the following format

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n$$

where

$$y_1(x) = J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Then we have

$$y_2(x) = J_0(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^n$$

$$\Downarrow$$

$$y_2'(x) = J_0'(x)\ln(x) + \frac{1}{x}J_0(x) + \sum_{n=1}^{\infty} nc_n^* x^{n-1}$$

$$\Downarrow$$

$$y_2''(x) = J_0''(x)\ln(x) + \frac{2}{x}J_0'(x) - \frac{1}{x^2}J_0(x) + \sum_{n=1}^{\infty} n(n-1)c_n^* x^{n-2}$$

Substituting the above equations into Bessel's equation of order 0, i.e.,

$$xy_2'' + y_2' + xy_2 = 0$$

we have

$$0 = xJ_0''(x)\ln(x) + 2J_0'(x) - \frac{1}{x}J_0(x) + \sum_{n=1}^{\infty} n(n-1)c_n^* x^{n-1}$$

$$+\ J_0'(x)\ln(x) + \frac{1}{x}J_0(x) + \sum_{n=1}^{\infty} nc_n^* x^{n-1}$$

$$+\ xJ_0(x)\ln(x) + \sum_{n=1}^{\infty} c_n^* x^{n+1}$$

$$\Downarrow$$

$$0 = \ln(x)\left[xJ_0''(x) + J_0'(x) + xJ_0(x)\right]$$

expression inside $[\cdot] = 0$ **because** $J_0(x)$ **is a solution**

$$+ \sum_{n=2}^{\infty} n(n-1)c_n^* x^{n-1} + \sum_{n=1}^{\infty} nc_n^* x^{n-1}$$

$$+ \sum_{n=1}^{\infty} c_n^* x^{n+1} + 2J_0'(x)$$

$$\Downarrow$$

$$2J_0'(x) + c_1^* + \sum_{n=2}^{\infty} n^2 c_n^* x^{n-1} + \sum_{n=1}^{\infty} c_n^* x^{n+1} = 0$$

Noting that

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-1} n!(n-1)!}$$

we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + c_1^* + \sum_{n=2}^{\infty} n^2 c_n^* x^{n-1} + \sum_{n=1}^{\infty} c_n^* x^{n+1} = 0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + c_1^* + 4c_2^* x + \sum_{n=3}^{\infty} n^2 c_n^* x^{n-1} + \sum_{n=3}^{\infty} c_{n-2}^* x^{n-1} = 0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{2^{2n-2} n!(n-1)!} + c_1^* + 4c_2^* x + \sum_{n=3}^{\infty} [n^2 c_n^* + c_{n-2}^*] x^{n-1} = 0$$

$$\Downarrow$$

$$c_1^* = 0$$

Also, note that the only even powers of x appearing on the left hand side of the equation occur in the last series, when n is odd.

The coefficient of these powers of x must be zero and hence must satisfy

$$n^2 c_n^* + c_{n-2}^* = 0, \quad n = 3, 5, 7, \cdots\cdots$$

$$\Downarrow$$

$$c_n^* = -\frac{1}{n^2} c_{n-2}^*, \quad n = 3, 5, 7, \cdots\cdots$$

i.e.,

$$c_3^* = -\frac{1}{9} c_1^* = 0$$

$$c_5^* = -\frac{1}{25} c_3^* = 0$$

$$c_7^* = -\frac{1}{49} c_5^* = 0 \cdots\cdots$$

Thus, in general,

$$c_{2m+1}^* = 0 \quad \text{for } m = 0, 1, 2, \cdots\cdots$$

We will now determine the even-indexed coefficients. First, we replace n by $2j$ in the second summation (note that $n^2 c_n^* - c_{n-2}^* = 0$ for $n = 3$. Thus, the second summation can start from $n = 4$, which implies that j can start from 2), and $n = j$ in the first summation, i.e.,

$$\sum_{j=1}^{\infty} \frac{(-1)^j x^{2j-1}}{2^{2j-2} j!(j-1)!} + 4c_2^* x + \sum_{j=2}^{\infty} [4j^2 c_{2j}^* + c_{2j-2}^*] x^{2j-1} = 0$$

$$\Downarrow$$

$$(4c_2^* - 1)x + \sum_{j=2}^{\infty} \left[\frac{(-1)^j}{2^{2j-2} j!(j-1)!} + 4j^2 c_{2j}^* + c_{2j-2}^* \right] x^{2j-1} = 0$$

Equating the coefficients of each power of x to zero, we have

$$c_2^* = \frac{1}{4}$$

$$c_{2j}^* = \frac{(-1)^{j+1}}{2^{2j} j^2 j!(j-1)!} - \frac{1}{4j^2} c_{2j-2}^*$$

With $j = 2$,

$$c_4^* = \frac{-1}{2^4 2^2 2} - \frac{1}{2^2 2^2 4} = \frac{-1}{2^2 4^2}\left[1 + \frac{1}{2}\right]$$

With $j = 3$,

$$c_6^* = \frac{1}{2^6 3^2 6 \cdot 2} + \frac{1 + \frac{1}{2}}{4 \cdot 3^2 2^2 4^2} = \frac{1}{2^2 4^2 6^2}\left[1 + \frac{1}{2} + \frac{1}{3}\right]$$

In general, we find that

$$c_{2j}^* = \frac{(-1)^{j+1}}{2^2 4^2 \cdots (2j)^2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}\right]$$

$$\Downarrow$$

$$c_{2j}^* = \frac{(-1)^{j+1}}{2^{2j}(j!)^2}\psi(j)$$

where

$$\psi(j) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}$$

A second solution of Bessel's equation of order zero may be written as

$$y_2(x) = J_0(x)\ln(x) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{2n}(n!)^2}\psi(n)x^{2n}$$

for $x > 0$.

Because of the logarithm term, this second solution is linearly independent from the first solution, $J_0(x)$.

Instead of $y_2(x)$ for a second solution, it is customary to use a particular linear combination of $J_0(x)$ and $y_2(x)$, denoted $Y_0(x)$ and defined by

$$Y_0(x) = \frac{2}{\pi}\left\{y_2(x) + \Big[\gamma - \ln(2)\Big]J_0(x)\right\}$$

in which γ is called Euler's constant and given by,

$$\gamma = \lim_{n\to\infty}[\psi(n) - \ln(n)] = 0.577215664901533\cdots$$

Since $Y_0(x)$ is a sum of constants times solutions of Bessel's equation of order 0, it is also a solution. Furthermore, $Y_0(x)$ is linearly independent from $J_0(x)$. Thus, the general solution of Bessel's equation of order 0 is given by

$$y(x) = \alpha_1 J_0(x) + \alpha_2 Y_0(x)$$

In terms of the series derived above for $y(x)$,

$$Y_0(x) = \frac{2}{\pi}\left\{J_0(x)\ln(x) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{2n}(n!)^2}\psi(n)x^{2n} + [\gamma - \ln(2)]J_0(x)\right\}$$

$$= \frac{2}{\pi}\left\{J_0(x)\left[\ln(\frac{x}{2}) + \gamma\right] + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{2n}(n!)^2}\psi(n)x^{2n}\right\}$$

$Y_0(x)$ is a Bessel's function of the second kind of order zero; with this choice of constants, $Y_0(x)$ is also called **Neumann's Function of Order Zero**.

A Second Solution of Bessel's Equation of Order v if v is a Positive Integer.

If v is a positive integer, say $v = k$, then a similar procedure as in the $k = 0$ case, but more involved calculation leads us to the following second solution of Bessel's equation of order $v = k$,

$$Y_k(x) = \frac{2}{\pi}\left\{J_k(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] - \sum_{n=0}^{k-1}\frac{(k-n-1)!}{2^{2n-k+1}n!}x^{2n-k}\right.$$

$$\left. + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}[\psi(n) + \psi(n+k)]}{2^{2n+k+1}n!(n+k)!}x^{2n+k}\right\}, \quad \psi(0) = 0$$

$Y_k(x)$ and $J_k(x)$ are linearly independent for $x > 0$, and the general solution of Bessel's equation of order k is given by

$$y(x) = \alpha_1 J_k(x) + \alpha_2 Y_k(x)$$

Although $J_k(x)$ is simple $J_v(x)$ for the case $v = k$, our derivation of $Y_k(x)$ does not suggest how $Y_v(x)$ might be defined if v is not a nonnegative integer. However, it is possible to define $Y_v(x)$, if v is not an integer, by letting

$$Y_v(x) = \frac{1}{\sin(v\pi)}\left[J_v(x)\cos(v\pi) - J_{-v}(x)\right]$$

This is a linear combination of $J_v(x)$ and $J_{-v}(x)$, two solutions of Bessel's equation of order v, and hence is also a solution of Bessel's equation of order v.

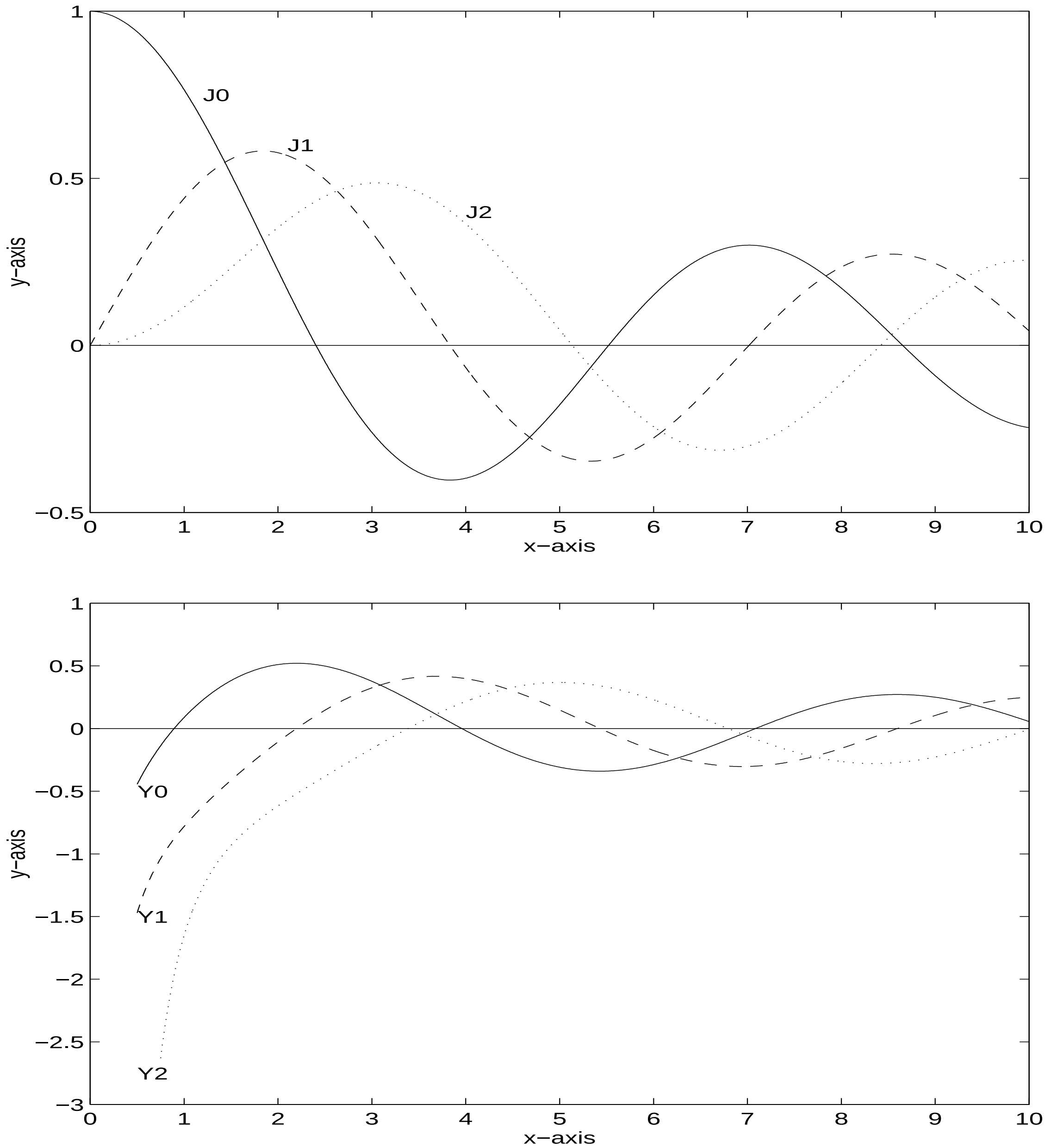
It can be shown (**very complicated!**) that one can obtain $Y_k(x)$, for k a nonnegative integer, from the above definition by taking the limit,

$$Y_k(x) = \lim_{v \to k} Y_v(x)$$

$Y_v(x)$ is called Neumann's Bessel's function of order v. It is linearly independent from $J_v(x)$ and hence the general solution of Bessel's equation of order v can be written as

$$y(x) = \alpha_1 J_v(x) + \alpha_2 Y_v(x)$$

which holds regardless v is an integer or not.



The Bessel Functions of the First and Second Kinds.

4. Modified Bessel Functions

Sometimes modified Bessel functions are encountered in modeling physical phenomena. First, note that

$$y(x) = \alpha_1 J_0(kx) + \alpha_2 Y_0(kx)$$

is the general solution of the differential equation

$$y'' + \frac{1}{x}y' + k^2 y = 0$$

Let $k = i$, where $i = \sqrt{-1}$, which implies $k^2 = i^2 = -1$. Then

$$y(x) = \alpha_1 J_0(ix) + \alpha_2 Y_0(ix)$$

is the general solution of

$$y'' + \frac{1}{x}y' - y = 0$$

This differential equation is called a modified Bessel's equation of order zero, and $J_0(ix)$ is a modified Bessel function of the first kind of order zero. Usually we denote

$$I_0(x) = J_0(ix)$$

Since $i^2 = -1$, substitution of ix for x in the series for J_0 yields:

$$I_0(x) = 1 + \frac{1}{2^2}x^2 + \frac{1}{2^2 4^2}x^4 + \frac{1}{2^2 4^2 6^2}x^6 + \cdots\cdots$$

Usually $Y_0(ix)$ is not used. Instead we use the function

$$K_0(x) = [\ln(2) - \gamma]I_0(x) - I_0(x)\ln(x) + \frac{1}{4}x^2 + \cdots\cdots$$

$K_0(x)$ is called a modified Bessel function of the second kind of order zero. The quantity γ is as usual the Euler's constant.

We now write the general solution of the differential equation

$$y'' + \frac{1}{x}y' - y = 0$$

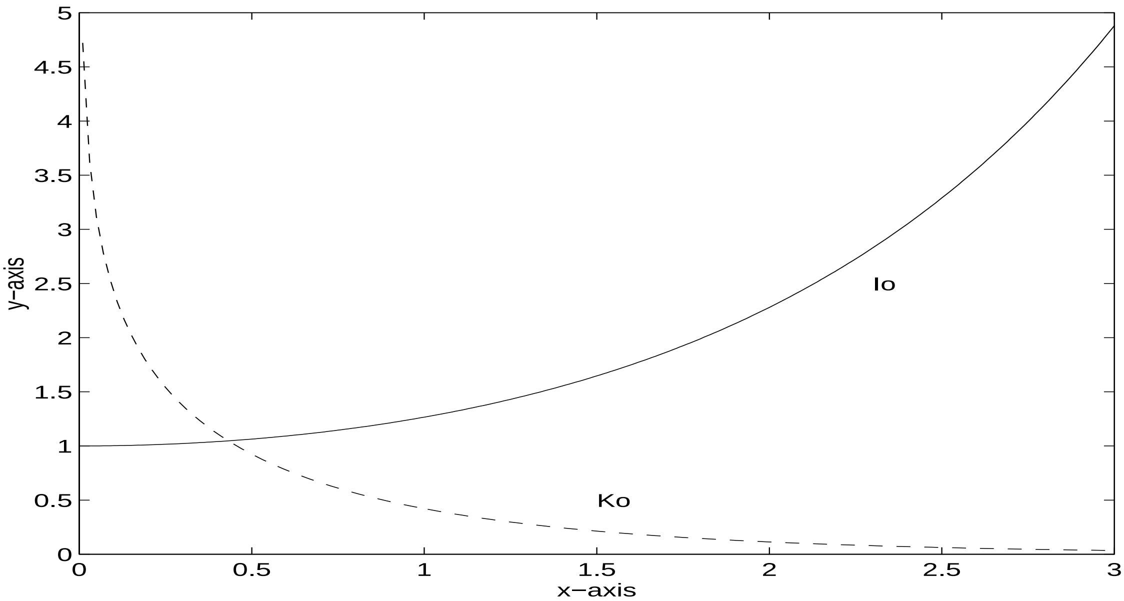
as

$$y(x) = \alpha_1 I_0(x) + \alpha_2 K_0(x)$$

Homework:

Show that the general solution of the differential equation

$$y'' + \frac{1}{x}y' - b^2 y = 0$$



The Modified Bessel Functions.

is given by

$$y(x) = \alpha_1 I_0(bx) + \alpha_2 K_0(bx)$$

Problem 21: (O'Neil, page 262)

Show that

$$[xI_0'(x)]' = xI_0(x)$$

Proof:

$$I_0(x) = J_0(ix)$$

$$\Downarrow$$

$$I_0'(x) = iJ_0'(ix)$$

$$\Downarrow$$

$$xI_0'(x) = ixJ_0'(ix)$$

$$\Downarrow$$

$$[xI_0'(x)]' = iJ_0'(ix) - xJ_0''(ix)$$

But $y = J_0(ix)$ is the solution of the modified Bessel equation

$$y'' + \frac{1}{x}y' - y = 0$$

or

$$xy'' + y' = xy \tag{4.1}$$

Also, note that

$$y = J_0(ix) \Longrightarrow y' = iJ_0'(ix) \Longrightarrow y'' = -J_0''(ix)$$

Substituting into (4.1) above, we have

$$-xJ_0''(ix) + iJ_0'(ix) = xJ_0(ix)$$

That is

$$[xI_0'(x)]' = xI_0(x)$$

Q.E.D.

Problem 10: (O'Neil page 262)

Show that $y_1(x) = x^a J_n(bx^c)$ and $y_2(x) = x^a Y_n(bx^c)$ are solutions of

$$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - n^2c^2}{x^2}\right) y = 0$$

for any constants a, b and c, and any nonnegative integer n.

Proof:

$$y_1 = x^a J_n(bx^c)$$

$$\Downarrow$$

$$y_1' = x^a J_n'(bx^c)bcx^{c-1} + ax^{a-1}J_n(bx^c) = bcx^{a+c-1}J_n'(bx^c) + ax^{a-1}J_n(bx^c)$$

$$\Downarrow$$

$$y_1'' = bcx^{a+c-1}J_n''(bx^c)bcx^{c-1} + bc(a+c-1)x^{a+c-2}J_n'(bx^c)$$

$$+ax^{a-1}J_n'(bx^c)bcx^{c-1} + a(a-1)x^{a-2}J_n(bx^c)$$

$$= b^2c^2x^{a+2c-2}J_n''(bx^c) + bc(2a+c-1)x^{a+c-2}J_n'(bx^c)$$

$$+a(a-1)x^{a-2}J_n(bx^c)$$

Substituting into the given differential equation, we get

$$b^2c^2\ x^{a+2c-2}J_n'' + bc(2a+c-1)x^{a+c-2}J_n' + a(a-1)x^{a-2}J_n$$

$$-\ (2a-1)bcx^{a+c-2}J_n' - a(2a-1)x^{a-2}J_n + a^2x^{a-2}J_n$$

$$+\ b^2c^2x^{a+2c-2}J_n - n^2c^2x^{a-2}J_n$$

$$= b^2c^2x^{a+2c-2}J_n'' + bc^2x^{a+c-2}J_n' + b^2c^2x^{a+2c-2}J_n - n^2c^2x^{a-2}J_n$$

$$= c^2x^{a-2}\Big\{b^2x^{2c}J_n'' + bx^cJ_n' + (b^2x^{2c} - n^2)J_n\Big\}$$

The factor inside $\{\cdots\}$ is equivalent to the left-hand side of the differential equation

$$z^2y'' + zy' + (z^2 - n^2)y = 0$$

when we substitute $z = bx^c$ and $y = J_n(z)$. Thus,

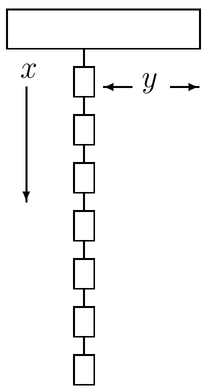
$$\Big\{b^2x^{2c}J_n'' + bx^cJ_n' + (b^2x^{2c} - n^2)J_n\Big\} = 0$$

That is: $y_1(x) = x^aJ_n(bx^c)$ is the solution of the given differential equation. For the other solution, $y_2(x)$, follow the same procedure above and try to do it yourself.

Q.E.D.

5. Applications of Bessel Functions

Displacement of a Suspended Chain



Suppose we have a heavy flexible chain. The chain is fixed at the upper end and free at the bottom.

We want to describe the oscillations caused by a small displacement in a vertical plane from the stable equilibrium position.

We will assume that each particle of the chain oscillates in a horizontal straight line.

Let m be the mass of the chain per unit length, L be the length of the chain, and $y(x,t)$ be the horizontal displacement at time t of the particle of chain whose distance from the point of suspension is x.

Consider an element of chain of length Δx. If the forces acting on the ends of this element are T and $T + \Delta T$, the horizontal component in Newton's Second Law of Motion (force equals to the rate of change of momentum with respect to time) is:

$$m(\Delta x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)\Delta x$$

(For those who are interested in the derivation of the above equation, please read Section 11.2 & 12.8 of the Reference Book by Wylie)

$$\Downarrow$$

$$m\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)$$

The weight of the chain below x where T acts, is:

$$T = mg(L - x)$$

Substituting into the above differential equation, we get

$$\frac{\partial^2 y}{\partial t^2} = -g\frac{\partial y}{\partial x} + g(L-x)\frac{\partial^2 y}{\partial x^2}$$

This is a partial differential equation. However, we can reduced the problem of solving it to a problem involving only an ordinary differential equation. Let $z = L - x$ and $u(z,t) = y(L - z, t)$. Then

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial y}{\partial x} = \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = -\frac{\partial u}{\partial z}$$

$$\frac{\partial^2 u}{\partial z^2} = -\frac{\partial}{\partial z}\left(\frac{\partial y}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\frac{\partial x}{\partial z} = \frac{\partial^2 y}{\partial x^2}$$

$$\Downarrow$$

$$\frac{\partial^2 u}{\partial t^2} = g\frac{\partial u}{\partial z} + gz\frac{\partial^2 u}{\partial z^2}$$

This is still a partial differential equation, which can be solved using p.d.e. method. Since we anticipate the oscillations to be periodic in t, we will attempt a solution of the form

$$u(z,t) = f(z)\cos(\omega t - \delta)$$

Substituting into the partial differential equation, we get

$$-\omega^2 f(z)\cos(\omega t - \delta) = gf'(z)\cos(\omega t - \delta) + gzf''(z)\cos(\omega t - \delta)$$

Dividing this equation by $gz\cos(\omega t-\delta)$, we get

$$f''(z)+\frac{1}{z}f'(z)+\frac{\omega^2}{gz}f(z)=0$$

Fortunately, we have shown that that

$$x^a J_n(bx^c) \quad \text{and} \quad x^a Y_n(bx^c)$$

are solutions of the differential equation

$$y''-\left(\frac{2a-1}{x}\right)y'+\left(b^2c^2x^{2c-2}+\frac{a^2-n^2c^2}{x^2}\right)y=0$$

Now, let

$$\begin{array}{rcl} 2a-1=-1 & \Longrightarrow & a=0 \\ 2c-2=-1 & \Longrightarrow & c=\dfrac{1}{2} \\ & \Downarrow & \\ b^2c^2=\dfrac{\omega^2}{g} & \Longrightarrow & b=\dfrac{2\omega}{\sqrt{g}} \\ & \Downarrow & \\ a^2-n^2c^2=0 & \Longrightarrow & n=0 \end{array}$$

Thus, the general solution is in terms of Bessel functions of order zero:

$$f(z)=\alpha_1 J_0\left(2\omega\sqrt{\frac{z}{g}}\right)+\alpha_2 Y_0\left(2\omega\sqrt{\frac{z}{g}}\right)$$

Now, from we know (see figure on page 22) that

$$Y_0\left(2\omega\sqrt{\frac{z}{g}}\right)\to-\infty$$

as $z\to 0^+$ (that is, as $x\to L$, i.e., at the bottom end of the chain). We must therefore choose $\alpha_2=0$ in order to have a bounded solution, as we expect from the physical setting of the problem. This leave us with

$$f(z)=\alpha_1 J_0\left(2\omega\sqrt{\frac{z}{g}}\right)$$

Thus,

$$u(z,t)=f(z)\cos(\omega t-\delta)=\alpha_1 J_0\left(2\omega\sqrt{\frac{z}{g}}\right)\cos(\omega t-\delta)$$

Hence,

$$y(x,t)=\alpha_1 J_0\left(2\omega\sqrt{\frac{L-x}{g}}\right)\cos(\omega t-\delta)$$

The frequencies of the normal oscillations of the chain are determined by using this general form of the solution for $y(x,t)$ together with condition that the upper end of the chain is fixed and therefore does not move. For all t, we must have

$$y(0,t)=0$$

Assume that $\alpha_1 \neq 0$. This requires that we have to choose ω such that

$$J_0\left(2\omega\sqrt{\frac{L}{g}}\right)=0$$

This gives values of ω which can be frequencies of the oscillations.

To find these admissible values of ω, we must consult a table of zeros of J_0. From a table of values of zeros of Bessel functions, we find that the first five positive solutions of $J_0(\alpha)=0$ are approximately 2.405, 5.520, 8.654, 11.792, 14.931. **(These values can also be found using commercially available software package such as MATLAB. Check out function `bessel` in MATLAB).** Using the these zeros, we obtain

$$
\begin{aligned}
2\omega_1\sqrt{\frac{L}{g}} = 2.405 &\implies \omega_1 = 1.203\sqrt{\frac{g}{L}} \\
2\omega_2\sqrt{\frac{L}{g}} = 5.520 &\implies \omega_2 = 2.760\sqrt{\frac{g}{L}} \\
2\omega_3\sqrt{\frac{L}{g}} = 8.645 &\implies \omega_3 = 4.327\sqrt{\frac{g}{L}} \\
2\omega_4\sqrt{\frac{L}{g}} = 11.792 &\implies \omega_4 = 5.896\sqrt{\frac{g}{L}} \\
2\omega_5\sqrt{\frac{L}{g}} = 14.931 &\implies \omega_5 = 7.466\sqrt{\frac{g}{L}}
\end{aligned}
$$

All these are admissible values of ω, and they represent approximate frequencies of the normal modes of oscillation.

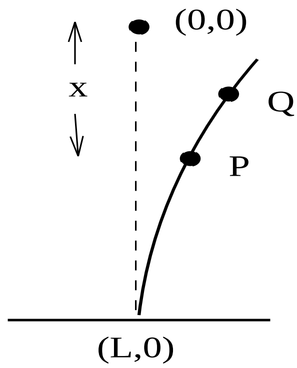
The approximate period T_j associated with ω_j is

$$T_j=\frac{2\pi}{\omega_j}$$

Special Remarks:

There are two features in the solution of this type of problems:

1. Change of variables was used to write solutions in terms of Bessel functions.



2. Much of information about the motion of the system was obtained from zeros of a Bessel function.

The Critical Length of a Vertical Rod

Suppose we have a thin elastic rod of uniform density. This rod is clamped in a vertical position Intuitively, if the rod is "too long" and the upper end is displaced slightly, the rod will remain in the displaced position after being released.

On the other hand, if the rod is "short enough", it will return to the vertical position after being released.

We would like to know where the transition occurs between being too long and short enough. That is, we want the **minimum length** at which the rod remains bent after being released. This length is called the **Critical Length** of the rod and of course will depend on the material of the rod.

To derive a mathematical model from which we can solve for this critical length, let L be the length of the rod and a be the radius of its cross-sectional circular. Let w be the weight per unit length and E be the Young's modulus for the rod. Note that E depends on the material of the rod. We should expect that this will influence the critical length. Finally, the moment of inertia about a diameter is $I = \pi a^4/4$.

Now, assume that the rod is in equilibrium and is displaced slightly from the vertical. The origin is as shown, and the x-axis is vertical, with downward as positive. Let $P(x, y)$ be a point on the rod, and let $Q(\xi, \eta)$ be a point slight above P.

The moment about P of the weight of an element $w\Delta x$ at Q is given by

$$w\Delta x[y(\xi) - y(x)]$$

Assume from the theory of elasticity the fact that the moment of the elastic forces about P is

$$EI\frac{d^2y}{dx^2}$$

Since the part of the rod above P is in equilibrium, we have

$$EI\frac{d^2y}{dx^2} = \int_0^x w[y(\xi) - y(x)]d\xi$$

Differentiate this equation with respect to x to get

$$EI\frac{d^3y}{dx^3} = w[y(x) - y(x)] - \int_0^x wy'(x)d\xi = -wx\frac{dy}{dx}$$

This yields a third order differential equation

$$EI\frac{d^3y}{dx^3} + wx\frac{dy}{dx} = 0$$

or

$$\frac{d^3y}{dx^3} + \frac{w}{EI}x\frac{dy}{dx} = 0$$

Let $u = \dfrac{dy}{dx}$ to obtain a second order differential equation:

$$\frac{d^2u}{dx^2} + \frac{w}{EI}xu = 0$$

Recall your tutorial problem (also Problem 1 on page 253 of O'neil): Use the fact that J_v is a solution of Bessel's equation of order v to show that $x^a J_v(bx^c)$ is a solution of the differential equation

$$y'' - \left(\frac{2a-1}{x}\right)y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - v^2c^2}{x^2}\right)y = 0$$

Compare with the above differential equation:

$$2a - 1 = 0 \implies a = \frac{1}{2}$$

$$\Downarrow$$

$$a^2 - v^2c^2 = 0 \implies v = \frac{1}{3}$$

$$\Uparrow$$

$$2c - 2 = 1; \implies c = \frac{3}{2}$$

$$\Downarrow$$

$$b^2c^2 = \frac{w}{EI} \implies b = \frac{2}{3}\sqrt{\frac{w}{EI}}$$

The general solution of the differential equation is given by

$$u = \frac{dy}{dx} = \alpha_1 x^{\frac{1}{2}} J_{\frac{1}{3}} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} x^{\frac{3}{2}} \right) + \alpha_2 x^{\frac{1}{2}} J_{-\frac{1}{3}} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} x^{\frac{3}{2}} \right)$$

There is no bending moment at the upper end of the rod. Thus,

$$\left. \frac{d^2 y}{dx^2} \right|_{x=0} = 0$$

This condition requires that we have to choose $\alpha_1 = 0$. Otherwise,

$$\begin{aligned}
&\left\{ \sqrt{x} J_{1/3} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} x^{3/2} \right) \right\}' \\
&= \left\{ \left(\frac{2}{3} \sqrt{\frac{w}{EI}} \right)^{-1/3} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} x^{3/2} \right)^{1/3} J_{1/3} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} x^{3/2} \right) \right\}' \\
&= \left(\frac{2}{3} \sqrt{\frac{w}{EI}} \right)^{-1/3} \left\{ t^{1/3} J_{1/3}(t) \right\}' \sqrt{\frac{w}{EI}} x^{1/2} \quad \textbf{(change from } x \textbf{ to } t\textbf{)} \\
&= \left(\frac{2}{3} \sqrt{\frac{w}{EI}} \right)^{-1/3} \left\{ t^{1/3} J_{\frac{1}{3}-1}(t) \right\} \sqrt{\frac{w}{EI}} x^{1/2} \quad \textbf{(see problem 3 on page 12)} \\
&= x J_{-\frac{2}{3}} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} x^{3/2} \right) \sqrt{\frac{w}{EI}} \to -\infty \quad \textbf{as } x \to 0^+
\end{aligned}$$

Therefore, we have to choose a solution having the following format

$$\frac{dy}{dx} = \alpha_2 \sqrt{x} J_{-1/3} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} x^{3/2} \right)$$

Furthermore, the lower end of the rod is clamped and so does not move; then

$$\left. \frac{dy}{dx} \right|_{x=L} = 0$$

In order to satisfy this condition with $\alpha_2 \neq 0$, we must have

$$J_{-1/3} \left(\frac{2}{3} \sqrt{\frac{w}{EI}} L^{3/2} \right) = 0$$

The critical length is the smallest positive number L which satisfies the above equation.

We find from the table or using software tools that the smallest positive zero of $J_{-1/3}$ is approximately 1.8663. Thus, the critical length L is determined approximately by

$$\frac{2}{3} \sqrt{\frac{w}{EI}} L^{2/3} = 1.8663$$

$$\Downarrow$$

$$L = \left(1.8663 \times \frac{3}{2}\sqrt{\frac{EI}{w}}\right)^{2/3} = 1.9863 \left(\frac{EI}{w}\right)^{1/3}$$

Alternating Current in a Circular Wire: the Skin Effect

Consider an alternating current of period $2\pi/\omega$, given by $D\cos(\omega t)$. Let R be the radius of the wire, ρ be its specific resistance, μ be the permeability, $x(r,t)$ be the current density at radius r and at time t, and $H(r,t)$ be the magnetic intensity at radius r from the centre of the wire at time t.

To derive an equation for the current density $x(r,t)$, we will need two laws of electromagnetic field theory. **Ampere's Law** states that the line integral of the electric force around a closed path equals to 4π times the integral of electric current through the path. **Faraday's Law** states that the line integral of the electric force around a closed path equals to the negative of the partial derivative with respect to time of the magnetic induction through the path.

Now consider a circular path of radius r within the wire, centred about the midpoint of the wire.

By Ampere's Law,

$$2\pi r H = 4\pi \int_0^r 2\pi r x dr$$

Differentiating with respect to r, we get

$$\frac{\partial}{\partial r}(rH) = 4\pi r x \tag{5.1}$$

Consider a closed rectangular path in the wire, with two sides along the axis of the cylinder and of length L, and the other sides of length r.

By Faraday's Law,

$$\rho L[x(0,t) - x(r,t)] = -\frac{\partial}{\partial t}\int_0^r \mu L H dr$$

Differentiating w.r.t. r, we get

$$\rho\frac{\partial x}{\partial r} = \mu\frac{\partial H}{\partial t} \tag{5.2}$$

We want to eliminate H from these equations to obtain an equation for x alone. To do that, we multiply Equation (5.2) by r and differentiate the resulting equation w.r.t. r. We obtain

$$\rho\frac{\partial}{\partial r}\left(r\frac{\partial x}{\partial r}\right) = \mu\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial t}\right) = \mu\frac{\partial}{\partial r}\left(\frac{\partial}{\partial t}[rH]\right) \tag{5.3}$$

because $\partial r/\partial t = 0$. Assume that we can reverse the order of the differentiation on the right-hand side of the above equation, we obtain

$$\rho\frac{\partial}{\partial r}\left(r\frac{\partial x}{\partial r}\right) = \mu\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}[rH]\right)$$

From Equation (5.1), we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial x}{\partial r}\right) = \frac{4\pi\mu}{\rho}\frac{\partial x}{\partial t} \tag{5.4}$$

To solve this equation, we will employ a device which is quite standard in mathematical treatments of electricity and magnetism. Let

$$z(r,t) = x(r,t) + iy(r,t)$$

and then replace x with z in Equation (5.4) to get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right) = \frac{4\pi\mu}{\rho}\frac{\partial z}{\partial t} \tag{5.5}$$

Now recall Euler's formula. We can write

$$\cos(\omega t) + i\sin(\omega t) = e^{i\omega t}$$

Anticipating periodic dependence of the current density on time, we will attempt a solution of Equation (5.5) of the form,

$$z(r,t) = f(r)e^{i\omega t}$$

Substituting this expression for z into Equation (5.5), we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rf'(r)e^{i\omega t}\right] = \frac{4\pi\mu}{\rho}i\omega f(r)e^{i\omega t}$$

$$\Downarrow$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rf'(r)\right] = i\frac{4\pi\mu\omega}{\rho}f(r)$$

$$\Downarrow$$

$$f''(r) + \frac{1}{r}f'(r) - i\frac{4\pi\mu\omega}{\rho}f(r) = 0 \tag{5.6}$$

Now let

$$k = \frac{1+i}{\sqrt{2}}\sqrt{\frac{4\pi\mu\omega}{\rho}} \quad \Longrightarrow \quad k^2 = i\frac{4\pi\mu\omega}{\rho}$$

Equation (5.6) becomes

$$f''(r) + \frac{1}{r}f'(r) - k^2 f(r) = 0$$

This is a modified Bessel's equation with general solution

$$f(r) = c_1 I_0(kr) + c_2 K_0(kr)$$

In order for $f(r)$ to remain finite as $r \to 0^+$, we must have $c_2 = 0$. Thus, $f(r)$ is of the form $c_1 I_0(kr)$, with I_0 the modified Bessel function of first kind of order zero. Then $z(r,t)$ has the form

$$z(r,t) = c_1 I_0(kr) e^{i\omega t}$$

The current density $x(r,t)$ is the real part of the above expression.

We have not yet used the initial assumption that the alternating current in the wire is given by $D\cos(\omega t)$.
Note that $D\cos(\omega t)$ is the real part of $De^{i\omega t}$. Since $De^{i\omega t}$ represents the total current, we have

$$De^{i\omega t} = \int_0^R 2\pi r z dr = 2\pi c_1 \int_0^R r I_0(kr) e^{i\omega t} dr$$

Upon dividing by $e^{i\omega t}$, we have

$$D = \int_0^R 2\pi r z dr = 2\pi c_1 \int_0^R r I_0(kr) dr$$

Recall that D is known. Thus, we obtain the constant c_1, which is given by

$$c_1 = \frac{D}{2\pi \int_0^R r I_0(kr) dr} \tag{5.7}$$

Note that (see e.g., page 23)

$$\Big[x I_0'(x)\Big]' = x I_0(x),$$

which implies

$$r I_0(kr) = \frac{r}{k^2} I_0'(kr)$$

and hence

$$\int_0^R r I_0(kr) dr = \frac{R}{k^2} I_0'(kR) \tag{5.8}$$

Substituting this into Equation (5.7), we get

$$c_1 = \frac{Dk^2}{2\pi R I_0'(kR)}$$

and hence

$$z(r,t) = \frac{Dk^2}{2\pi R I_0'(kR)} I_0(kr) e^{i\omega t}$$

The current density $x(r,t)$ is the real part of the expression. Note that both k and k^2 are not real numbers. There is no simple way to separate the above expression into real and imaginary parts.

The Skin Effect:

We will now apply this analysis to a mathematically derivation of the skin effect:

It has been observed that, for sufficiently high frequencies, the current flowing through a circular wire at radius r is small compared with the total current, even for r nearly equal to R. This means that "most" of the current in a cylindrical wire flows through a thin layer at the outer surface, i.e., at the "skin" of the wire.

To derive this effect from the model, begin with the solution for z. The total current through a cylinder of radius r is given by

$$\int_0^r 2\pi z(r,t)dr = \frac{Dk^2}{2\pi R I_0'(kR)} \int_0^r 2\pi r I_0(kr)e^{i\omega t}dr$$

$$= \frac{Dk^2}{R I_0'(kR)} e^{i\omega t} \int_0^r r I_0(kr)dr$$

By Equation (5.8), with r in place of R we have

$$\int_0^r r I_0(kr)dr = \frac{r}{k^2} I_0'(kr)$$

Thus, the total current through a cylinder of radius r is

$$\underbrace{De^{i\omega t}}_{\textbf{total current in the wire}} \times \underbrace{\frac{r}{R}\frac{I_0'(kr)}{I_0'(kR)}}_{\textbf{ratio of the current in cylinder of radius } r \textbf{ to the total current}}$$

We want to know this ratio behaves for large k, which is proportional to $\sqrt{\omega}$. It can be shown that for large x, $I_0(x)$ can be approximated by

$$I_0(x) \approx \frac{Ae^x}{\sqrt{x}}$$

for some constant A. Hence, for large x,

$$I_0'(x) \approx \frac{Ae^x}{\sqrt{x}} + Ae^x\left(-\frac{1}{2}x^{-3/2}\right) = \frac{Ae^x}{\sqrt{x}}\left(1 - \frac{1}{2x}\right) \approx \frac{Ae^x}{\sqrt{x}}$$

Thus, if ω is large, which implies that k is large in magnitude, we have

$$\frac{r}{R}\frac{I_0'(kr)}{I_0'(kR)} \approx \frac{r}{R}\frac{e^{kr}}{\sqrt{r}}\frac{\sqrt{R}}{e^{kR}} = \sqrt{\frac{r}{R}}\ e^{-k(R-r)}$$

Given any $r < R$, $e^{-k(R-r)}$ can be made as small in magnitude as we like by choosing appropriate (large) ω (or k). Thus, for large k, the ratio of the current in the cylinder of radius r to the total current $\to 0$, and this is the so-called skin effect.

In fact, we can choose r as close as we like to R, and this conclusion continues to hold, for sufficiently large frequencies.

$\diamond\diamond\diamond$

6. Legendre's Equation and Legendre Polynomials

The differential equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

in which α is a constant, is called *Legendre's Equation.*

It occurs in a variety of problems involving quantum mechanics, astronomy and analysis of heat conduction, and is often seen in settings in which it is natural to use spherical coordinates.

Write Legendre's equation as

$$y'' - \left\{\frac{2x}{1-x^2}\right\} y' + \left\{\frac{\alpha(\alpha+1)}{1-x^2}\right\} y = 0$$

The coefficient functions are analytic at every point except $x = 1$ and $x = -1$. In particular, both functions have Maclaurin series expansions in $(-1, 1)$.

Since zero is an ordinary point of Legendre's equation, there are two linearly independent solutions which are analytic in $(-1, 1)$ and which can be found by the power series method. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Substituting into Legendre's Equation, we have

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \alpha(\alpha+1)a_n x^n = 0$$

The first summation can be written as

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} \alpha(\alpha+1)a_n x^n = 0$$

Combining summations from $n = 2$ onwards and writing terms for $n = 0$ and $n = 1$ separately, we have

$$\Big[2a_2 + \alpha(\alpha+1)a_0\Big] + \Big[6a_3 - 2a_1 + \alpha(\alpha+1)a_1\Big] x$$

$$+ \sum_{n=2}^{\infty} \Big\{(n+2)(n+1)a_{n+2} - [n^2 + n - \alpha(\alpha+1)]a_n\Big\} x^n = 0$$

The coefficient of each power of x on the left-hand side of the equation must be zero, i.e.,

$$2a_2 + \alpha(\alpha+1)a_0 = 0 \implies a_2 = -\frac{\alpha(\alpha+1)}{2}a_0 \tag{6.1}$$

$$6a_3 - 2a_1 + \alpha(\alpha+1)a_1 = 0 \implies a_3 = -\frac{(\alpha-1)(\alpha+2)}{6}a_1 \tag{6.2}$$

and for $n = 2, 3, 4, \cdots$

$$(n+2)(n+1)a_{n+2} - [n^2 + n - \alpha(\alpha+1)]a_n = 0$$

The last equation is the recurrence relation. Re-write

$$\begin{aligned}[n^2 + n - \alpha(\alpha+1)] &= n^2 - \alpha^2 + n - \alpha \\ &= (n-\alpha)(n+\alpha) + (n-\alpha) \\ &= (n-\alpha)(n+\alpha+1)\end{aligned}$$

Thus,

$$a_{n+2} = -\frac{(\alpha-n)(n+\alpha+1)}{(n+2)(n+1)}a_n$$

for $n = 2, 3, 4, \cdots$. In view of (6.1) and (6.2), we see that in fact the recurrence relation is also valid for $n = 0, 1$. That is

$$a_{n+2} = -\frac{(\alpha-n)(n+\alpha+1)}{(n+2)(n+1)}a_n, \quad \text{for } n = 0, 1, 2, 3, \cdots$$

The above recurrence relation expresses a_{n+2} as a multiple of a_n. Thus, a_2 is a multiple of a_0; a_4 is a multiple of a_2 (hence of a_0); and so on, with every even-indexed coefficient a multiple of a_0. Similarly, every odd-indexed coefficient is a multiple of a_1.

For the even-indexed coefficients, we have

$$a_2 = -\frac{(\alpha+1)\alpha}{1\cdot 2}a_0$$

$$\begin{aligned}a_4 &= -\frac{(\alpha+3)(\alpha-2)}{3\cdot 4}a_2 \\ &= +\frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{4!}a_0\end{aligned}$$

$$a_6 = -\frac{(\alpha+5)(\alpha+3)(\alpha+1)\alpha(\alpha-2)(\alpha-4)}{6!}a_0$$

In general, we have

$$a_{2n} = (-1)^n \frac{(\alpha+2n-1)(\alpha+2n-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2n+2)}{(2n)!}\, a_0$$

and

$$a_{2n+1} = (-1)^n \frac{(\alpha+2n)(\alpha+2n-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)}{(2n+1)!}\, a_1$$

We can obtain two linearly independent solutions of Legendre's equation by making choice for a_0 and a_1.

If we choose $a_0 = 1$ and $a_1 = 0$, we get one solution:

$$y_1(x) = 1+\sum_{n=1}^{\infty}(-1)^n \frac{(\alpha+2n-1)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2n+2)}{(2n)!} x^{2n}$$

$$= 1 - \frac{(\alpha+1)\alpha}{2}x^2 + \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{24}x^4$$

$$- \frac{(\alpha+5)(\alpha+3)(\alpha+1)\alpha(\alpha-2)(\alpha-4)}{720}x^6 + \cdots$$

By letting $a_0 = 0$ and $a_1 = 1$, we obtain a second solution:

$$y_2(x) = x+\sum_{n=1}^{\infty}(-1)^n \frac{(\alpha+2n)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2n+1)}{(2n+1)!} x^{2n+1}$$

$$= x - \frac{(\alpha+2)(\alpha-1)}{6}x^3 + \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{120}x^5 - \cdots$$

These power series converge for all $x \in (-1, 1)$.

Note that for some α, one or the other of these series solutions is a polynomial. For example, if $\alpha = 2$, then $a_4 = 0$; and hence $a_6 = a_8 = \cdots = 0$. Thus, y_1 is just a second degree polynomial, i.e.,

$$y_1(x) = 1 - 3x^2$$

If $\alpha = 3$, then $a_5 = 0$; so $a_7 = a_9 = \cdots = 0$; and hence y_2 is a third degree polynomial, i.e.,

$$y_2(x) = x - \frac{5}{3}x^3$$

In fact, whenever α is a nonnegative integer, the power series for either $y_1(x)$ (if α is even) or $y_2(x)$ (if α is odd) reduces to a finite series, and we obtain a polynomial solution of Legendre's equation. Such polynomial solutions are useful in many applications, including methods for approximating solutions of equations $f(x) = 0$.

In such applications, it is helpful to standardize specific polynomial solutions so that their values can be tabulated. The convention is to multiply y_1 or y_2 for each term by a constant which makes the value of the polynomial 1 at $x = 1$.

The resulting polynomials are called Legendre polynomials and are denoted by $P_n(x)$, i.e., $P_n(x)$ is the solution of Legendre's equation with $\alpha = n$.

The first few Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1, & P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), & P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), & P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \end{aligned}$$

Although these polynomials are defined for all x, they are solutions of Legendre's equation only for $-1 < x < 1$ and for appropriate α.

It can be shown that α must be chosen as a nonnegative integer in order to obtain nontrivial solutions of Legendre's equation which are bounded on $[-1, 1]$. This is particularly important in models of phenomena in physics and engineering, where boundedness of the solution is a natural expectation.

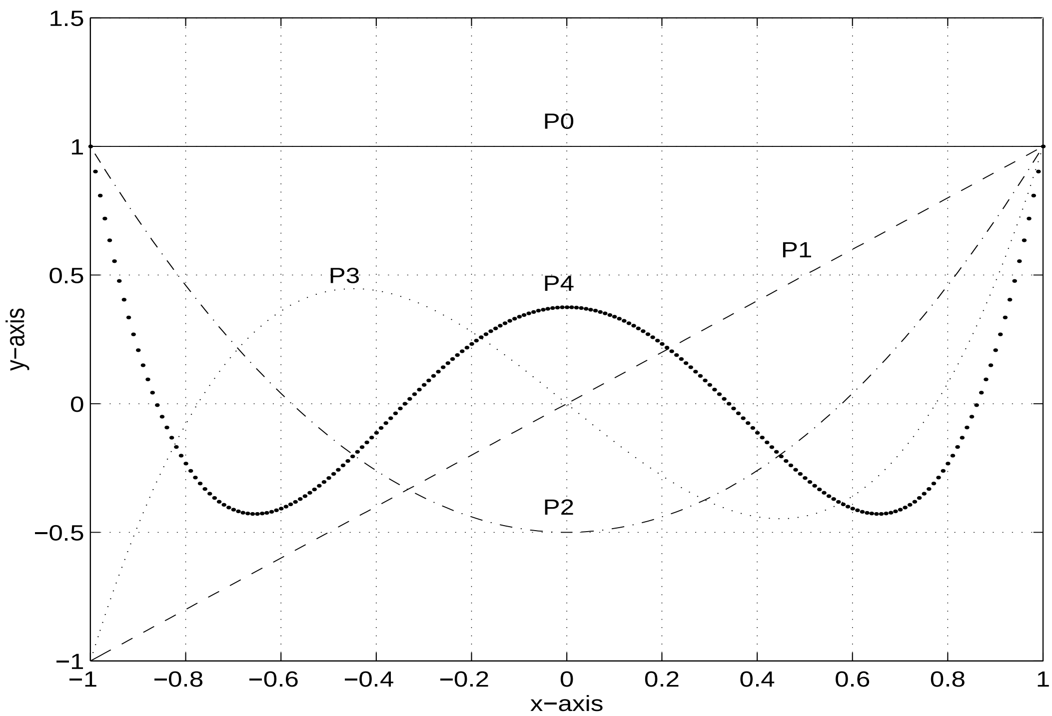
Theorem 6.1. *If m and n are distinct nonnegative integer, then*

$$\int_{-1}^{1} P_m(x) P_n(x) dx = 0$$

Proof: P_n and P_m are solutions of Legendre's differential equations with $\alpha = n$ and $\alpha = m$, respectively. Hence, we have

$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$$

$$(1 - x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0$$



The Legendre Polynomials Over Interval $(-1, 1)$.

Multiplying the 1st equation by P_m and the 2nd equation by P_n, we obtain

$$(1-x^2)P_n''P_m - 2xP_n'P_m + n(n+1)P_nP_m = 0$$

$$-\ (1-x^2)P_m''P_n - 2xP_m'P_n + m(m+1)P_mP_n = 0$$

$$= (1-x^2)(P_n''P_m - P_m''P_n) - 2x(P_n'P_m - P_m'P_n)$$

$$+ [n(n+1) - m(m+1)]P_mP_n = 0$$

The above equation can ne written as

$$(1-x^2)\frac{d}{dx}[P_n'P_m - P_m'P_n] - 2x[P_n'P_m - P_m'P_n] = [(m(m+1) - n(n+1)]P_mP_n$$

$$\Downarrow$$

$$\frac{d}{dx}\Big\{(1-x^2)[P_n'P_m - P_m'P_n]\Big\} = [m(m+1) - n(n+1)]P_mP_n$$

Integrating

$$[m(m+1) - n(n+1)]\int_{-1}^{1} P_m(x)P_n(x)dx = \{(1-x^2)[P_n'P_m - P_m'P_n]\}\Big|_{-1}^{1} = 0$$

Since $m \neq n$, we have

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0$$

Q.E.D.

Exercise Problems: (Problem 2 O'Neil pg 237): Show that

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!x^{n-2k}}{2^n k!(n-k)!(n-2k)!}$$

in which $[n/2]$ = largest integer $\leq n/2$. Use this formula to compute $P_0(x)$ through $P_5(x)$.

(Problem 3 O'Neil pg 237): Rodrigue's formula states that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

for $n = 1, 2, 3, \cdots$. Prove this formula, assuming the formula for $P_n(x)$ from Problem 2. Use Rodrigue's formula to compute $P_0(x)$ through $P_5(x)$.

7. Properties of the Legendre Polynomials

Generating Function for Legendre Polynomials

The generating function for Legendre Polynomials is

$$P(x,r) = (1 - 2xr + r^2)^{-1/2}$$

To see why this is called a generating function function, recalled the binomial expansion

$$(1-z)^{-1/2} = 1 + \frac{1}{2}z + \frac{1}{2!}\frac{1}{2}\frac{3}{2}z^2 + \frac{1}{3!}\frac{1}{2}\frac{3}{2}\frac{5}{2}z^3 + \cdots$$

If we let $z = 2xr - r^2$, we have

$$P(x,r) = 1 + \frac{1}{2}(2xr - r^2) + \frac{3}{8}(2xr-r^2)^2 + \frac{5}{16}(2xr-r^2)^3 + \cdots$$

Rewrite this series in ascending powers of r to give

$$P(x,r) = 1 + xr + \left(-\frac{1}{2} + \frac{3}{2}x^2\right)r^2 + \left(-\frac{3}{2}x + \frac{5}{2}x^3\right)r^3 + \cdots$$

$$= P_0(x) + P_1(x)r + P_2(x)r^2 + P_3(x)r^3 + \cdots$$

Thus, the coefficient of r^n in the series for $P(x,r)$ is exactly $P_n(x)$, i.e.,

$$P(x,r) = \sum_{n=0}^{\infty} P_n(x)r^n$$

Theorem 7.1. (Recurrence Relation for Legendre Polynomial). *For each positive integer n,*

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

for all $-1 \le x \le 1$.

Proof: Differentiating the generating function

$$P(x,r) = (1 - 2xr + r^2)^{-1/2}$$

w.r.t. r, we obtain

$$\frac{\partial P}{\partial r} = -\frac{1}{2}(1 - 2xr + r^2)^{-3/2}(-2x + 2r)$$

$$= (1 - 2xr + r^2)^{-3/2}(x - r)$$

$$\Downarrow$$

$$(1 - 2xr + r^2)\frac{\partial P}{\partial r} = (1 - 2xr + r^2)^{-1/2}(x - r)$$

$$\Downarrow$$

$$(1 - 2xr + r^2)\frac{\partial P}{\partial r} - (x - r)P(x,r) = 0$$

Note that from the property of the generating function, i.e.,

$$P(x,r) = \sum_{n=0}^{\infty} P_n(x) r^n$$

We have

$$\frac{\partial P}{\partial r} = \sum_{n=0}^{\infty} nP_n(x) r^{n-1} = \sum_{n=1}^{\infty} nP_n(x) r^{n-1}$$

$$\Downarrow$$

$$(1 - 2xr + r^2)\sum_{n=1}^{\infty} nP_n(x) r^{n-1} - (x - r)\sum_{n=0}^{\infty} P_n(x) r^n = 0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} nP_n(x) r^{n-1} - \sum_{n=1}^{\infty} 2xnP_n(x) r^n + \sum_{n=1}^{\infty} nP_n(x) r^{n+1} - \sum_{n=0}^{\infty} xP_n(x) r^n$$

$$+ \sum_{n=0}^{\infty} P_n(x) r^{n+1} = 0$$

$$\Downarrow$$

$$\sum_{n=0}^{\infty}(n+1)P_{n+1}(x)r^n + \sum_{n=1}^{\infty}(-2xn)P_n(x)r^n + \sum_{n=2}^{\infty}(n-1)P_{n-1}(x)r^n$$

$$- \sum_{n=0}^{\infty} xP_n(x)r^n + \sum_{n=1}^{\infty} P_{n-1}(x)r^n = 0$$

$$\Downarrow$$

$$0 = P_1(x) + 2P_2(x)r - 2xP_1(x)r - xP_0(x) - xP_1(x)r + P_0(x)r$$

$$+ \sum_{n=2}^{\infty} \{(n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x) - xP_n(x) + P_{n-1}(x)\} r^n$$

$$\Downarrow$$

$$\Big[P_1(x) - xP_0(x)\Big] + \Big[(1+1)P_2(x) - (2+1)xP_1(x) + P_0(x)\Big] r$$

$$+ \sum_{n=2}^{\infty}\Big\{(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x)\Big\} r^n = 0$$

$$\Downarrow$$

$$P_1(x) - xP_0(x) = 0$$

$$(1+1)P_2(x) - (2+1)xP_1(x) + P_0(x) = 0$$

and, for $n = 2, 3, 4, \cdots$

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

This completes the proof of the recurrence relation. **Q.E.D.**

Theorem 7.2. *The coefficient of x^n in $P_n(x)$ is*

$$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$$

Proof: Let c_n be the coefficient of x^n in $P_n(x)$, and consider the recurrence relation

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

The coefficient of x^{n+1} in $(n+1)P_{n+1}(x)$ is equal to $(n+1)c_{n+1}$. The coefficient of x^{n+1} in $-(2n+1)xP_n(x)$ is equal to $-(2n+1)c_n$. There is no other x^{n+1} term in the recurrence relation. Thus the coefficient of x^{n+1} is

$$(n+1)c_{n+1} - (2n+1)c_n = 0$$

$$\Downarrow$$

$$c_{n+1} = \frac{2n+1}{n+1} c_n$$

Working backwards, we have

$$\begin{aligned} c_n &= \frac{2n-1}{n} c_{n-1} \\ &= \frac{2n-1}{n} \frac{2n-3}{n-1} c_{n-2} \\ &= \frac{2n-1}{n} \frac{2n-3}{n-1} \frac{2n-5}{n-2} c_{n-3} \\ &= \cdots \\ &= \frac{(2n-1)(2n-3)(2n-5)\cdots(1)}{n(n-1)(n-2)\cdots(1)} c_0 \end{aligned}$$

But c_0 is the coefficient of x^0 in $P_0(x) = 1$. $\Rightarrow c_0 = 1$

$$\Downarrow$$

$$c_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$$

Q.E.D.

Theorem 7.3. *For each positive integer n,*

$$nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0$$

Proof: Given the generating function:

$$P(x,r) = (1 - 2xr + r^2)^{-1/2}$$

Differentiating it w.r.t. x, we obtain

$$\frac{\partial P}{\partial x} = -\frac{1}{2}(-2r)(1 - 2xr + r^2)^{-3/2}$$

$$\Downarrow$$

$$(1 - 2xr + r^2)\frac{\partial P}{\partial x} = r(1 - 2xr + r^2)^{-1/2}$$

$$\Downarrow$$

$$(1 - 2xr + r^2)\frac{\partial P}{\partial x} - rP(x,r) = 0$$

$$\Downarrow$$

$$(1 - 2xr + r^2) = \Big\{ rP(x,r) \Big\} / \frac{\partial P}{\partial x} \tag{7.1}$$

Recall the earlier result by differentiating $P(x, r)$ w.r.t. r, i.e.,

$$(1 - 2xr + r^2)\frac{\partial P}{\partial r} - (x - r)P(x,r) = 0 \tag{7.2}$$

Equations (7.1) and (7.2) implies that

$$r\frac{\partial P}{\partial r} - (x - r)\frac{\partial P}{\partial x} = 0$$

Note that by the property of the generating function, i.e.,

$$P(x,r) = \sum_{n=0}^{\infty} P_n(x) r^n$$

we have

$$\frac{\partial P}{\partial r} = \sum_{n=1}^{\infty} nP_n(x) r^{n-1}$$

$$\frac{\partial P}{\partial x} = \sum_{n=0}^{\infty} P_n'(x) r^n$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} nP_n(x) r^n - \sum_{n=0}^{\infty} xP_n'(x) r^n + \sum_{n=0}^{\infty} P_n'(x) r^{n+1} = 0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} nP_n(x) r^n - \sum_{n=0}^{\infty} xP_n'(x) r^n + \sum_{n=1}^{\infty} P_{n-1}'(x) r^n = 0$$

$$\Downarrow$$

$$-xP_0'(x) + \sum_{n=1}^{\infty} \Big\{ nP_n(x) - xP_n'(x) + P_{n-1}'(x) \Big\} r^n = 0$$

Hence (because the fact that $P_0(x) = 1$)

$$nP_n(x) - xP_n'(x) + P_{n-1}'(x) = 0$$

Q.E.D.

Theorem 7.4. *For each positive integer n,*

$$nP_{n-1}(x) - P_n'(x) + xP_{n-1}'(x) = 0$$

Proof: We had in the previous proof the following equality

$$rP(x,r) = (1 - 2xr + r^2)\frac{\partial P}{\partial x}$$

$$r\frac{\partial P}{\partial r} - (x - r)\frac{\partial P}{\partial x} = 0$$

$$\Downarrow$$

$$r\frac{\partial P}{\partial r} = (x - r)\frac{\partial P}{\partial x}$$

Note that

$$r\frac{\partial}{\partial r}[rP(x,r)] = rP + r^2\frac{\partial P}{\partial r}$$

$$\Downarrow$$

$$r\frac{\partial}{\partial r}[rP(x,r)] = (1 - 2xr + r^2)\frac{\partial P}{\partial x} + r(x - r)\frac{\partial P}{\partial x} = (1 - rx)\frac{\partial P}{\partial x}$$

$$\Downarrow$$

$$r\frac{\partial}{\partial r}[rP(x,r)] - (1 - rx)\frac{\partial P}{\partial x} = 0$$

Also note that

$$P(x,r) = \sum_{n=0}^{\infty} P_n(x)r^n$$

$$\Downarrow$$

$$\frac{\partial P}{\partial x} = \sum_{n=0}^{\infty} P_n'(x)r^n$$

and

$$rP(x,r) = \sum_{n=0}^{\infty} P_n(x)r^{n+1}$$

$$\Downarrow$$

$$r\frac{\partial}{\partial r}[rP(x,r)] = \sum_{n=0}^{\infty}(n+1)P_n(x)r^{n+1}$$

$$\Downarrow$$

$$\begin{aligned} 0 &= r\frac{\partial}{\partial r}[rP] - (1 - rx)\frac{\partial P}{\partial x} \\ &= \sum_{n=0}^{\infty}(n+1)P_n(x)r^{n+1} - \sum_{n=0}^{\infty} P_n'(x)r^n + \sum_{n=0}^{\infty} xP_n'(x)r^{n+1} \end{aligned}$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty} nP_{n-1}(x)r^n - \sum_{n=0}^{\infty} P_n'(x)r^n + \sum_{n=1}^{\infty} xP_{n-1}'(x)r^n = 0$$

$$\Downarrow$$

$$\sum_{n=1}^{\infty}\Big\{nP_{n-1}(x) - P_n'(x) + xP_{n-1}'(x)\Big\}r^n - P_0'(x) = 0$$

$$\Downarrow$$

$$nP_{n-1}(x) - P_n'(x) + xP_{n-1}'(x) = 0$$

Q.E.D.

Orthogonal Polynomials

We have shown that if m and n are distinct nonnegative integers, then

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0$$

In view of this, we can say that the Legendre polynomials are *Orthogonal* to each other on the interval $[-1, 1]$. We also say that the Legendre polynomials form a set of *Orthogonal Polynomials* on the interval on $[-1, 1]$.

The orthogonal property can be used to write many functions as series of Legendre polynomials. This will be important in solving certain boundary value problems in partial differential equations.

For now, we will see how to write any polynomial as such a series. Let $q(x)$ be a polynomial of degree m. We will see how to choose numbers α_0, α_1, α_2, $\cdots$, α_m such that

$$q(x) = \sum_{k=0}^{m} \alpha_k P_k(x) \qquad \text{for } -1 \le x \le 1$$

Multiply the above equation by $P_j(x)$, where j is any integer from 0 to m inclusive, i.e.,

$$q(x)P_j(x) = \alpha_0 P_0(x)P_j(x) + \alpha_1 P_1(x)P_j(x) + \cdots + \alpha_m P_m(x)P_j(x)$$

Integrating both sides of the above equation from -1 to 1, we have

$$\int_{-1}^{1} q(x)P_j(x)dx = \alpha_0 \int_{-1}^{1} P_0(x)P_j(x)dx + \alpha_1 \int_{-1}^{1} P_1(x)P_j(x)dx + \cdots + \alpha_m \int_{-1}^{1} P_m(x)P_j(x)dx \qquad (7.3)$$

Because of the relation

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{for } m \ne n$$

$$\Downarrow$$

$$\int_{-1}^{1} P_k(x)P_j(x)dx$$

on the right hand side of the equation (7.3) is zero, except for that one integral in which $k = j$.

This leaves us with

$$\int_{-1}^{1} q(x)P_j(x)dx = \alpha_j \int_{-1}^{1} [P_j(x)]^2 dx$$

$$\Downarrow$$

$$\alpha_j = \frac{\int_{-1}^{1} q(x)P_j(x)dx}{\int_{-1}^{1} [P_j(x)]^2 dx} \qquad \text{for } j = 0, 1, 2, \cdots m$$

These numbers can be computed because we know each $P_j(x)$ and we are given $q(x)$. Using these numbers, we can write $q(x)$ as a series of Legendre polynomials.

For example,

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

$$x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$1 - 4x^2 + 2x^3 = P_0(x) - 4\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] + 2\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right]$$

$$= -\frac{1}{3}P_0(x) + \frac{6}{5}P_1(x) - \frac{8}{3}P_2(x) + \frac{4}{5}P_3(x)$$

Any Polynomial can be Written as a Finite Series of Legendre Polynomials.

Theorem 7.5. *Let m and n be nonnegative integers, with $m < n$. Let $q(x)$ be any polynomial of degree m. Then*

$$\int_{-1}^{1} q(x)P_n(x)dx = 0$$

That is, the integral, from -1 to 1, of a Legendre polynomial multiplied by any polynomial of lower degree, is zero.

Proof: We have shown that for any polynomial $q(x)$ of degree m, there exist scalar α_0, α_1, $\cdots$, α_m such that

$$q(x) = \alpha_0 P_0(x) + \alpha_1 P_1(x) + \cdots + \alpha_m P_m(x)$$

Then

$$\int_{-1}^{1} q(x)P_n(x)dx = \alpha_0 \int_{-1}^{1} P_0(x)P_n(x)dx + \alpha_1 \int_{-1}^{1} P_1(x)P_n(x)dx$$

$$+ \cdots + \alpha_m \int_{-1}^{1} P_m(x)P_n(x)dx$$

and each of the integrals on the right hand side is zero by the property that

$$\int_{-1}^{1} P_j(x)P_n(x)dx = 0$$

for $j = 0, 1, 2, \cdots, m < n.$ **Q.E.D.**

Theorem 7.6.

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{2n+1}$$

for $n = 0, 1, 2, \cdots$

Proof: Let c_n be coefficient of x^n in $P_n(x)$ and also let the coefficient of x^{n-1} in $P_{n-1}(x)$ be c_{n-1}. Define

$$q(x) = P_n(x) - \frac{c_n}{c_{n-1}} x P_{n-1}(x)$$

The x^n term in $P_n(x)$ is cancelled by the x^n term in $-\frac{c_n}{c_{n-1}} x P_{n-1}(x)$. Thus, $q(x)$ has degree $n-1$ or lower.

We therefore have

$$P_n(x) = \frac{c_n}{c_{n-1}} x P_{n-1}(x) + q(x)$$

in which $q(x)$ has degree less than or equal to $n-1$.

$$\Downarrow$$

$$[P_n(x)]^2 = P_n(x)\left[\frac{c_n}{c_{n-1}} x P_{n-1}(x) + q(x)\right] = \frac{c_n}{c_{n-1}} x P_{n-1}(x)P_n(x) + q(x)P_n(x)$$

Integrating this equation from -1 to 1, we have

$$\int_{-1}^{1} [P_n(x)]^2 dx = \frac{c_n}{c_{n-1}} \int_{-1}^{1} x P_{n-1}(x)P_n(x)dx + \int_{-1}^{1} q(x)P_n(x)dx$$

$$= \frac{c_n}{c_{n-1}} \int_{-1}^{1} x P_{n-1}(x)P_n(x)dx + 0$$

Now use the recurrence relation,

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

$$\Downarrow$$

$$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$$

$$\Downarrow$$

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{c_n}{c_{n-1}}\int_{-1}^{1}\frac{n+1}{2n+1}P_{n-1}(x)P_{n+1}(x)dx$$

$$+ \; \frac{c_n}{c_{n-1}}\int_{-1}^{1}\frac{n}{2n+1}[P_{n-1}(x)]^2dx$$

$$= 0 + \frac{c_n}{c_{n-1}}\int_{-1}^{1}\frac{n}{2n+1}[P_{n-1}(x)]^2dx$$

$$\Downarrow$$

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{c_n}{c_{n-1}}\frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2dx$$

Recall from Theorem 7.2 that

$$c_n = \frac{1\cdot 3\cdot \cdots \cdot (2n-1)}{n!}$$

Then

$$c_{n-1} = \frac{1\cdot 3\cdot \cdots \cdot (2n-3)}{(n-1)!}$$

$$\Downarrow$$

$$\frac{c_n}{c_{n-1}} = \frac{2n-1}{n}$$

$$\Downarrow$$

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{2n-1}{n}\frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2dx = \frac{2n-1}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2dx$$

We can work backwards,

$$\int_{-1}^{1}[P_n(x)]^2dx = \frac{2n-1}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2dx$$

$$= \frac{(2n-1)(2n-3)}{(2n+1)(2n-1)}\int_{-1}^{1}[P_{n-2}(x)]^2dx$$

$$= \frac{(2n-1)(2n-3)(2n-5)}{(2n+1)(2n-1)(2n-3)}\int_{-1}^{1}[P_{n-3}(x)]^2dx$$

$$= \cdots\cdots\cdots$$

$$= \frac{(2n-1)(2n-3)(2n-5)\cdots 3\cdot 1}{(2n+1)(2n-1)(2n-3)\cdots 5\cdot 3}\int_{-1}^{1}[P_0(x)]^2dx$$

$$= \frac{1}{2n+1} \int_{-1}^{1} [P_0(x)]^2 dx$$

$$= \frac{2}{2n+1}$$

Q.E.D.

Finally, we can write any polynomial of degree m as a finite series of Legendre polynomials,

$$q(x) = \sum_{k=0}^{m} \alpha_k P_k(x) \qquad \textbf{for } -1 \le x \le 1$$

with

$$\alpha_k = \frac{\int_{-1}^{1} q(x) P_k(x) dx}{\int_{-1}^{1} [P_k(x)]^2 dx} = \frac{2k+1}{2} \int_{-1}^{1} q(x) P_k(x) dx$$

for $k = 0, 1, 2, \cdots m$**.**

◇◇◇

8. Boundary Value Problems in Partial Differential Equations

A partial differential equation is an equation containing one or more partial derivatives, e.g.,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

We seek a solution $u(x,t)$ which depends on the independent variables x and t.

A solution of a partial differential equation is a function which satisfies the equation.

For example,

$$u(x,t) = \cos(2x)e^{-4t}$$

is a solution of the above mentioned differential equation since

$$\frac{\partial u}{\partial t} = -4\cos(2x)e^{-4t}$$

$$\frac{\partial^2 u}{\partial x^2} = -4\cos(2x)e^{-4t}$$

Occasionally, a partial differential equation may be solved by inspection. For example, consider

$$\frac{\partial u}{\partial x} = -4\frac{\partial u}{\partial y}$$

There may be many solutions of this equation, but we can guess one by reasoning as follows:

If $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ were both constants, we could find a solution easily. Try

$$u(x,y) = Ax + By$$

with A and B constants. Then

$$\frac{\partial u}{\partial x} = A \qquad \text{and} \qquad \frac{\partial u}{\partial y} = B$$

Substituting them into the partial differential equation, we find that the proposed $u(x,y)$ is a solution for the given differential equation if and only if

$$A = -4B$$

Therefore, any function defined by

$$u(x,y) = B(-4x + y)$$

with B being a constant, is a solution of the partial differential equation.

Order of a Partial Differential Equation

A partial differential equation (p.d.e.) is said to be of order n if it contains an nth order partial derivative but none of higher order. For example, the following so-called Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is of order 2. The p.d.e.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^5 u}{\partial t^5} - \frac{\partial u}{\partial t}$$

is of order 5.

Linear Case

The general linear first order p.d.e. in two variables (with u as a function of the independent variables x and y) is

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + f(x,y)u + g(x,y) = 0$$

The general second order linear p.d.e. in two variables has the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x}$$
$$+ e(x,y)\frac{\partial u}{\partial y} + f(x,y)u + g(x,y) = 0$$

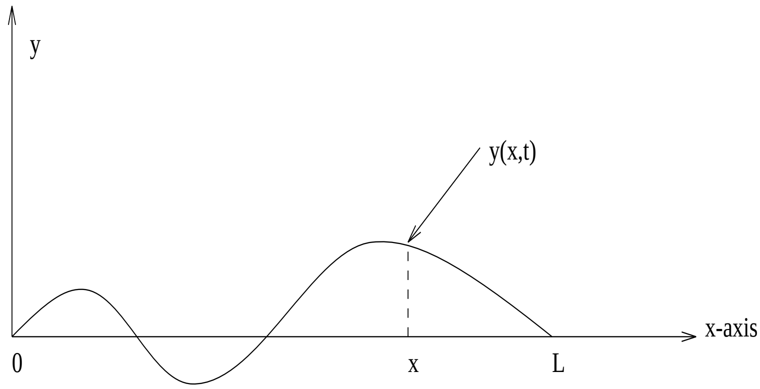
Most of the equations we encounter will be in one of these two forms.

In both cases, the equation is said to be *homogeneous* if $g(x,y) = 0$ for all (x,y) under consideration and *nonhomogeneous* if $g(x,y) \neq 0$ for some (x,y).

We will devote most of our time to equations governing *vibration* and *heat conduction* phenomena.

Main Tools:

Fourier series, integrals, transforms and Laplace transform.



9. The Wave Equation

Suppose we have a flexible elastic string stretched between two pegs. We want to describe the ensuing motion if the string is lifted and then released to vibrate in a vertical plane.

Place the x-axis along the length of the string at the rest. At any time t and horizontal coordinate x, let $y(x,t)$ be the vertical displacement of the string.

We want to determine equations which will enable us to solve for $y(x,t)$, thus obtaining a description of the shape of the string at any time.

We will begin by modelling a simplified case. Neglect damping forces such as air resistance and the weight of the string and assume that the tension $T(x,t)$ in the string always acts tangentially to the string. Assume that the string can only move in the vertical direction, i.e., the horizontal component of the tension is a constant. Also assume that the mass ρ per unit length is a constant.

Applying Newton's 2nd law of motion to the segment of the string between x and $x+\Delta x$, we have

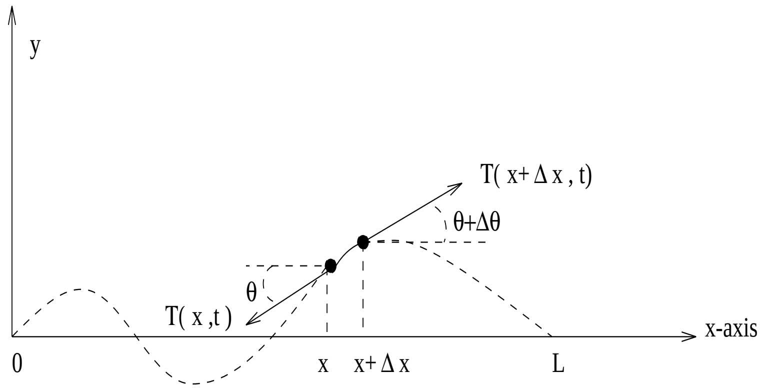
$$\begin{aligned}\textbf{Net force due to tension} &= \textbf{Segment mass} \\ &\quad \times\ \textbf{Acceleration of the centre of mass of the segment}\end{aligned}$$

For small Δx, consideration of the vertical component of the equation gives us approximately:

$$T(x+\Delta x,t)\sin(\theta+\Delta\theta) - T(x,t)\sin\theta = \rho\Delta x \frac{\partial^2 y}{\partial t^2}(\overline{x},t)$$

where $\overline{x}$ is the centre of the segment in x-axis.

$$\Downarrow$$



$$\frac{T(x+\Delta x,t)\sin(\theta+\Delta\theta)-T(x,t)\sin\theta}{\Delta x}=\rho\frac{\partial^2 y}{\partial t^2}(\overline{x},t)$$

As a convenience, we write

$$v(x,t)=T(x,t)\sin\theta$$

i.e., the vertical component of the tension. Hence we have

$$\frac{v(x+\Delta x,t)-v(x,t)}{\Delta x}=\rho\frac{\partial^2 y}{\partial t^2}(\overline{x},t)$$

Let $\Delta x\to 0$, then $\overline{x}\to x$, and we obtain

$$\frac{\partial v}{\partial x}=\rho\frac{\partial^2 y}{\partial t^2} \tag{9.1}$$

Write $h(x,t)=T(x,t)\cos\theta$, i.e., the horizontal component of the tension at (x,t), then

$$v(x,t)=h(x,t)\tan\theta=h(x,t)\frac{\partial y}{\partial x}$$

Substitute this for v in Equation (9.1):

$$\frac{\partial}{\partial x}\left[h\frac{\partial y}{\partial x}\right]=\rho\frac{\partial^2 y}{\partial t^2}$$

To compute the partial derivative on the left, recall that the horizontal component of the tension of the segment is constant; i.e.,

$$h(x+\Delta x,t)-h(x,t)=0$$

Therefore h is independent of x; hence

$$\frac{\partial}{\partial x}\left[h\frac{\partial y}{\partial x}\right]=h\frac{\partial^2 y}{\partial x^2}$$

Finally, we have

$$h\frac{\partial^2 y}{\partial x^2}=\rho\frac{\partial^2 y}{\partial t^2}$$

Let $a^2 = \dfrac{h}{\rho}$, we obtain a so-called 1-D wave equation,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The motion of the string will be influenced by both the initial position and the initial velocity of the string. Therefore we must specify initial conditions:

$$y(x,0) = f(x) \qquad \textbf{initial position}$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \qquad \textbf{initial velocity}$$

with $f(x)$ and $g(x)$ given functions defined on $[0, L]$. The initial conditions must hold for $0 \le x \le L$.

Next, we consider the boundary conditions. Since the ends of the string are fixed, we have

$$y(0,t) = y(L,t) = 0 \qquad t \ge 0$$

The wave equation, together with initial and boundary conditions is an example of a **boundary value** problem.

To be more clear, we can put all of them together, i.e.,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ t > 0)$$

$$y(0,t) = y(L,t) = 0 \qquad (t > 0)$$

$$y(x,0) = f(x) \qquad (0 < x < L)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \qquad (0 < x < L)$$

We expect on physical grounds that this problem will have a unique solution.

We can also include in the model additional forces acting on the string. For example, if an external force of magnitude F units per unit length acts parallel to the y axis, the wave equation must be adjusted by addition of a term F/ρ, i.e.,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} + \frac{1}{\rho} F \qquad 0 < x < L, \ \ t > 0$$

Note that if F is the weight of the string, then set $F = -g$ in the above p.d.e..

In two dimensions, we might have a membrane covering region R in the plane and fixed on a frame forming the boundary R. The membrane is set in motion, with vibrations occuring vertical to the plane of the membrane.

If $z(x, y, t)$ is the vertical coordinate at time t of the particle at point (x, y) in the membrane, the p.d.e. for z is:

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left[\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right]$$

for (x, y) in R. This is the 2-D wave equation.

To determine z uniquely, we must first include conditions, which specify the initial positions and velocity of the membrane.

$$z(x, y, 0) = f(x, y) \qquad \text{for } (x, y) \in R$$

$$\frac{\partial z}{\partial t}(x, y, 0) = g(x, y) \qquad \text{for } (x, y) \in R$$

Finally, the condition that the membrane is fixed to the frame means that points on the border of the membrane do not move, i.e.,

$$z(x, y, t) = 0$$

for all $t > 0$ and (x, y) on the boundary of R.

10. The Heat Equation

This is to study temperature distribution in a straight, thin bar under simple circumstances. Suppose we have a straight, thin bar of constant density ρ and constant cross-sectional area A placed along the x-axis from 0 to L.

Assume that the sides of the bar are insulated and so not allow heat loss and that the temperature on the cross-section of the bar perpendicular to the x-axis at x is a function $u(x, t)$ of x and t, independent of y.

Let the specific heat of the bar be c, and let the thermal conductivity be k, both constant. Now consider a typical segment of the bar between $x = \alpha$ and $x = \beta$.

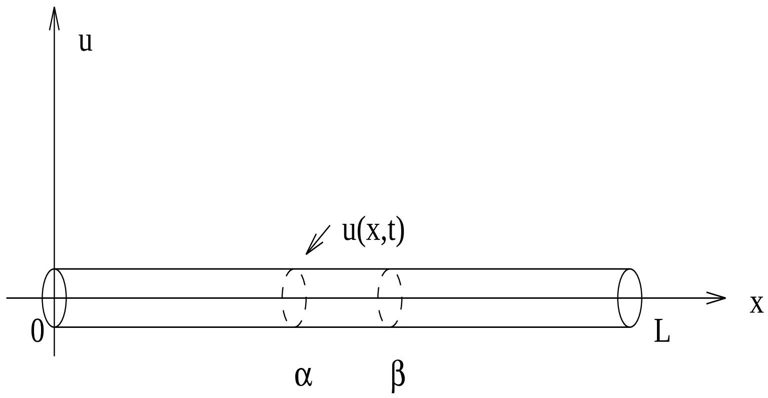
By the definition of specific heat, the rate at which heat energy accumulates in this segment of the bar is:

$$\int_\alpha^\beta c\rho A \frac{\partial u}{\partial t} dx$$

By Newton's law of cooling, heat energy flows within this segment from the warmer to the cooler end at a rate equal to k times the negative of the temperature gradient.

Therefore, the net rate at which heat energy enters the segment of bar between α and β at time t is:

$$kA \frac{\partial u}{\partial x}(\beta, t) - kA \frac{\partial u}{\partial x}(\alpha, t)$$



In the absence of heat production within the segment, the rate at which heat energy accumulates within the segment must balance the rate at which heat energy enters the segment. Hence,

$$\int_{\alpha}^{\beta} c\rho A \frac{\partial u}{\partial t} dx = kA\frac{\partial u}{\partial x}(\beta, t) - kA\frac{\partial u}{\partial x}(\alpha, t)$$

Note that the right hand side of the above equation can be written as:

$$kA\frac{\partial u}{\partial x}(\beta, t) - kA\frac{\partial u}{\partial x}(\alpha, t) = kA\int_{\alpha}^{\beta} \frac{\partial^2 u}{\partial x^2} dx$$

Therefore the whole equation can be written in the following form:

$$\int_{\alpha}^{\beta} \left[c\rho A \frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2} \right] dx = 0$$

This must hold for every α and β with $0 \le \alpha < \beta \le L$. If

$$c\rho A \frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2}$$

were nonzero for any $t > 0$ and some $x_0 \in [0, L]$, we could choose an interval $[\alpha, \beta]$ about x_0 in which

$$c\rho A \frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2}$$

is strictly positive or strictly negative, and we would then have

$$\int_{\alpha}^{\beta} \left[c\rho A \frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2} \right] dx \ne 0$$

This is a contradiction.

Thus, we conclude that

$$c\rho A \frac{\partial u}{\partial t} - kA\frac{\partial^2 u}{\partial x^2} = 0$$

for all $x \in [0, L]$ and $t > 0$.

This is the heat equation, which is more customary written as:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where $a^2 = k/(c\rho)$ is called the thermal diffusivity of the bar.

To determine u uniquely, we need boundary conditions (information at the ends of the bar) and initial conditions (temperature throughout the bar at time zero). The p.d.e., together with these pieces of information, constitutes a boundary value problem for the temperature function u.

Example 1:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, \ \ t > 0)$$

$$u(0,t) = u(L,t) = T \quad (t > 0)$$

$$u(x,0) = f(x) \quad (0 < x < L)$$

This boundary value problem models the temperature distribution in a bar of length L, whose ends are kept at constant temperature T, and the initial temperature in the cross-section at x is a given function $f(x)$.

Example 2:

A perfectly insulated bar, in which we replace the boundary conditions

$$u(0,t) = u(L,t) = T$$

of the first example with the conditions

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \quad (t > 0)$$

i.e., there is no heat flows across the ends of the bar.

Example 3:

Free radiation (convection): The bar loses heat by radiation from the ends into the surrounding medium, which is assumed to be maintained at a constant temperature T_a. In this case, the boundary conditions are

$$\frac{\partial u}{\partial x}(0,t) = A[u(0,t) - T_a]$$

$$\frac{\partial u}{\partial x}(L,t) = -A[u(L,t) - T_a]$$

for $t > 0$ with A being a positive constant. Note that if the bar is hotter than the surrounding medium, the heat flow (change in temperature per unit length, or $\frac{\partial u}{\partial x}$) must be positive at the left end of the bar and negative at the right end.

We can also have a combination of these different types of conditions occurring in a boundary value problem. For example, suppose we have a bar with its left end maintained at a constant temperature T_1 and its right end radiating into a medium of temperature T_2, with an initial temperature distribution given by $f(x)$. The boundary value problem modelling the above temperature distribution is:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, \ \ t > 0)$$

$$u(0,t) = T_1 \quad (t > 0)$$

$$\frac{\partial u}{\partial x}(L,t) = -A[u(L,t) - T_2] \quad (t > 0)$$

$$u(x,0) = f(x) \qquad (0 < x < L)$$

In two dimensions, the heat equation is:

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right]$$

In three dimensions,

$$\frac{\partial u}{\partial t} = a^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

Corresponding boundary and initial conditions must be specified to determine unique solutions of these partial differential equations.

11. Laplace's Equation; Poisson's Equation; Dirichlet Problem and Neumann Problem

Laplace's Equation

Laplace's equation in 2-D:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Laplace's equation in 3-D:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Laplace's equation is also called the steady-state heat equation, i.e., it is the heat equation when

$$\frac{\partial u}{\partial t} = 0$$

It has important applications in *heat conduction, fluid flow, study of electrical field potentials.*

It is often written as

$$\nabla^2 u = 0$$

A function satisfying $\nabla^2 u = 0$ is called a *harmonic function.*

Dirichlet Problem

The Dirichlet problem is to find a function which is harmonic in a given set M and takes on predetermined values on the boundary of M. For example, in 3-D, we could have the boundary value problem

$$\nabla^2 u = 0 \qquad \text{in } M$$

$$u(x, y, z) = f(x, y, z) \qquad \text{for } (x, y, z) \text{ on } \Sigma,$$

where Σ is a piecewise-smooth surface bounding M and f is a given function.

A typical Dirichlet problem in the 2-D plane would be:

$$\nabla^2 u = 0 \qquad \text{in } D$$

$$u(x, y) = g(x, y) \text{ for } (x, y) \text{ on } C$$

where C is a piecewise-smooth curve bounding the set D in the plane.

Neumann Problem

A Nuemann problem in 3-D consists of finding a function u such that $\nabla^2 u = 0$ for (x, y, z) in a region M, subject to the condition that the normal derivative of u takes on prescribed values on the boundary of the region, i.e.,

$$\nabla^2 u = 0 \qquad \text{in } M$$

$$\frac{\partial u}{\partial \eta} = f(x, y, z) \quad \text{for } (x, y, z) \text{ on } \Sigma$$

where Σ is the surface bounding M and $\frac{\partial u}{\partial \eta}$ denotes the directional derivative in the direction of the normal to Σ.

Poisson's Equation (Applicable to Potential for Electrical Fields)

The following equation is called Poisson's equation,

$$\nabla^2 u = f$$

with f being a given function. When f is identically zero, Poisson's equation becomes Laplace's equation.

Laplace's Equation in Cylindrical and Spherical Coordinates

Cylindrical coordinates (r, θ, z)

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

where

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{if } x \neq 0$$

Here the value of θ is determined by the signs of x and y. For example, if x and y are both negative, (x, y) is in the 3rd quadrant, so we must have $\pi < \theta < 3\pi/2$, even though the ratio y/x is positive.

Now suppose that u is a function of (x, y, z) and that u and its first and second partial derivatives are continuous throughout some set M of 3-D space.

Since x, y and z are functions of r, θ and z, we may think of u also as a function of r, θ an dz.

By the chain rule,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} \\ &= \frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}} + \frac{\partial u}{\partial \theta}\left(-\frac{y}{x^2+y^2}\right) + 0 \\ &= \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta} \qquad (11.1)\end{aligned}$$

By a similar calculation,

$$\frac{\partial u}{\partial y} = \frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta} \qquad (11.2)$$

To calculate second derivatives, differentiate equation (11.1) with respect to x:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) - \frac{\partial u}{\partial \theta}\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) + \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) - \frac{y}{r^2}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \theta}\right)$$

Applying the chain rule again, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{r^3}\frac{\partial u}{\partial r} + \frac{2xy}{r^4}\frac{\partial u}{\partial \theta} + \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{r^3}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{y^2}{r^4}\frac{\partial^2 u}{\partial \theta^2}$$

If we differentiate equation (11.2) with respect to y, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3}\frac{\partial u}{\partial r} - \frac{2xy}{r^4}\frac{\partial u}{\partial \theta} + \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{r^3}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{x^2}{r^4}\frac{\partial^2 u}{\partial \theta^2}$$

$$\Downarrow$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\Downarrow$$

Laplace's equation in cylindrical coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

For Laplace's equation in polar coordinates, simply omit the z-dependency:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Spherical Coordinates

Spherical coordinates are related to rectangular coordinates by

$$x = \rho\cos(\theta)\sin(\varphi), \quad y = \rho\sin(\theta)\sin(\varphi), \quad z = \rho\cos(\varphi)$$

where ρ is the magnitude of the vector $P = (x, y, z)$, and θ and φ are respectively the angles from x-axis and z-axis to P. The Laplacian in spherical coordinates is then given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2\sin^2(\varphi)}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\cot(\varphi)}{\rho^2}\frac{\partial u}{\partial \varphi}$$

The Laplace's equation in spherical coordinates is $\nabla^2 u = 0$.

12. Fourier Series Solution of the Wave Equation

Recall the wave equation of an initially displaced vibrating string with zero initial velocity,

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L, \;\; t > 0)$$

$$y(0,t) = y(L,t) = 0 \qquad (t > 0)$$

$$y(x,0) = f(x); \qquad (0 < x < L)$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \qquad (0 < x < L)$$

This boundary value problem models the vibration of an elastic string of length L, fastened at the ends, picked up at time zero to assume the shape of the graph of $y = f(x)$ and released from rest.

The Fourier method or method of separation of variables is to find a solution of the form

$$y(x,t) = X(x)T(t) \tag{12.1}$$

with appropriate $X(x)$ and $T(t)$ that solves the above mentioned boundary value problem.

At time t, $y(x,t)$ is the vertical displacement of the particle of string having coordinate x. We attempt a solution of (12.1). Substitute it into the p.d.e. to get

$$XT'' = a^2 X'' T$$

$$\Downarrow$$

$$\frac{X''}{X} = \frac{T''}{a^2 T}$$

We have "separated" x and t; the left hand side is a function of x alone, and the right hand side is a function of t. Since x and t are independent, we can fix the right hand side by choosing $t = t_0$, and the left hand side must be equal to

$$\frac{T''(t_0)}{a^2 T(t_0)} \qquad \text{for all } x \text{ in } (0, L)$$

Thus $\dfrac{X''}{X}$ must be constant. But the $\dfrac{T''}{a^2 T}$ must be equal to the same constant.

Denote this constant $-\lambda$ (it is also called *separation constant*), then

$$\frac{X''}{X} = \frac{T''}{a^2 T} = -\lambda$$

$$\Downarrow$$

$$X'' + \lambda X = 0$$

$$T'' + \lambda a^2 T = 0$$

These are two ordinary differential equations for X and T.

Next, look at the boundary conditions for $y(x,t)$ and relate them to X and T. From the condition that the left end of the string is fixed, we have

$$y(0,t) = X(0)T(t) = 0$$

for $t > 0$. Since $T(t)$ cannot be zero for all $t > 0$ (if the string is to move), $X(0) = 0$. Similarly,

$$y(L,t) = X(L)T(t) = 0$$

for $t > 0$ implies that $X(L) = 0$.

Next, initial condition

$$\frac{\partial y}{\partial t}(x,0) = 0$$

requires that

$$X(x)T'(0) = 0$$

for $0 < x < L$. Therefore $T'(0) = 0$.

At this point, we have two problems for X and T, namely

$$X'' + \lambda X = 0$$

$$X(0) = X(L) = 0$$

and

$$T'' + \lambda a^2 T = 0$$

$$T'(0) = 0$$

A value for λ for which the above problem, either the problem associated with X or T, has a nontrivial solution (nonzero at some points) is called eigenvalue of this problem. For such a λ, any nontrivial solution for X or for T is called eigenfunction.

We will consider different cases on λ. We assume that λ is real, as we expect from the physics of the problem.

Case 1: $\lambda = 0$

Then $X'' = 0$, so $X(x) = cx + d$ for some constants c and d. Then the condition $X(0) = 0$ implies $d = 0$ and $X(L) = cL = 0$ implies $c = 0$.

$$\Downarrow$$

The solution $X(x) = 0$ for $0 \le x \le L$, and we will have $y(x,t) = 0$ as the solution. This is the case if $f(x) = 0$ for $0 \le x \le L$ because then the string was not displaced initially and simply remain stationary. If, however, $f(x) \ne 0$ for some x, $X(x)$ cannot be identically zero, and we must discard this case.

Case 2: $\lambda < 0$.

For this case, we write $\lambda = -k^2$ with $k > 0$. The equation for X is the given by

$$X'' - k^2 X = 0$$

with general solution

$$X(x) = ce^{kx} + de^{-kx}$$

Since $X(0) = 0 = c + d \Rightarrow d = -c$ and

$$X(x) = c(e^{kx} - e^{-kx}) = 2c\sinh(kx)$$

Then

$$X(L) = 2c\sinh(kL) = 0$$

Since k and L are both positive, $\sinh(kL) > 0$, and we conclude that $c = 0$ and hence $d = -c = 0$.

As in Case 1, we obtain $X(x) = 0$ for $0 \le x \le L$. This case does not yield a solution unless again the string was not moved initially.

Case 3: $\lambda > 0$

We can write $\lambda = k^2$ with $k > 0$. Then

$$X'' + k^2 X = 0$$

has a general solution

$$X(x) = c\cos(kx) + d\sin(kx)$$

Since $X(0) = c = 0$, $\Rightarrow X(x) = d\sin(kx)$. We also require that $X(L) = d\sin(kL) = 0$. In order to choose $d \ne 0$, we must have $\sin(kL) = 0$. This holds if kL is positive integer multiple of π, i.e.,

$$kL = n\pi, \quad n = 1, 2, 3, \cdots$$

(Recall that k and L are positive). Then

$$\lambda = k^2 = \frac{n^2\pi^2}{L^2}$$

for $n = 1, 2, 3, \cdots$. Corresponding to each positive integer n, we therefore have a solution for X:

$$X_n(x) = d_n \sin\left(\frac{n\pi}{L}x\right)$$

Now look at the problem for T: With $\lambda = \dfrac{n^2\pi^2}{L^2}$, we have

$$T'' + \frac{n^2\pi^2a^2}{L^2}T = 0\ ; \quad T'(0) = 0$$

The general solution for T is given by

$$T(t) = \alpha\cos\left(\frac{n\pi a}{L}t\right) + \beta\sin\left(\frac{n\pi a}{L}t\right)$$

Since

$$T'(0) = \beta(n\pi a/L) = 0 \qquad \Longrightarrow \qquad \beta = 0$$

For each positive integer n, we have a solution for T:

$$T_n(t) = \alpha_n\cos\left(\frac{n\pi a}{L}t\right)$$

We now have, for each positive integer n, a function

$$y_n(x,t) = X_n(x)T_n(t) = A_n\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

in which $A_n = d_n\alpha_n$ is a constant yet to be determined.

Each of these functions satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}; \quad (0 < x < L,\ t > 0)$$

together with the boundary conditions

$$y(0,t) = y(L,t) = 0$$

for $t > 0$. The y_n's also satisfy one initial condition

$$\frac{\partial y}{\partial t}(x,0) = 0 \quad (0 < x < L)$$

We must choose n and A_n to satisfy the remaining condition,

$$y(x,0) = f(x)$$

Depending on $f(x)$, this may be possible. For example, if

$$f(x) = 8\sin(5\pi x/L)$$

we can choose $n = 5$ and $A_n = 8$. The solution of the boundary value problem for this initial displacement function f is

$$y(x,t) = 8\sin(5\pi x/L)\cos(5\pi at/L)$$

However, if $f(x)$ is NOT a constant multiple of a sine function, we cannot choose any one integer n and constant A_n so that $y(x,0) = f(x)$.

In this event, we attempt an infinite superposition of the y_n's and write

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

The condition $y(x,0) = f(x)$ requires that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}x\right)$$

Note that this equation is the Fourier sine expansion of $f(x)$ on $[0, L]$. We should choose the A_n's as the Fourier coefficients in this expansion, i.e.,

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

With this choice of the constants, we have the formal solution

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(\xi)\sin\left(\frac{n\pi}{L}\xi\right)d\xi\right]\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

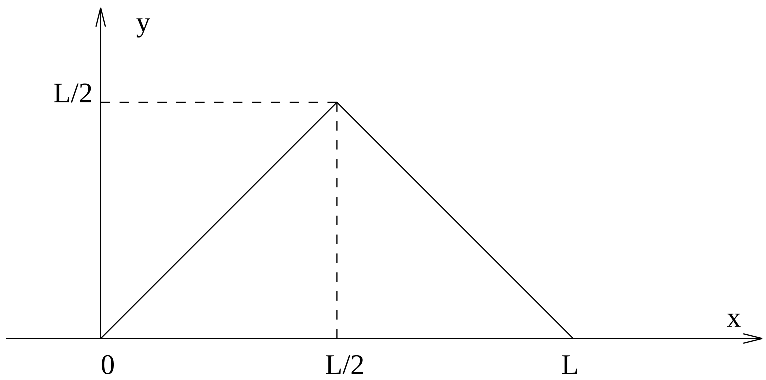
The integral in $[\cdots\cdots]$ is part of the Fourier coefficients, and ξ is used as the dummy variable of integration to avoid confusion with x.

Example

Suppose that initially the string is picked up $L/2$ units at its centre point and then release from the rest.

The initial position function is

$$f(x) = \begin{cases} x & 0 \le x \le L/2 \\ L - x & L/2 \le x \le L \end{cases}$$



Then we have

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx \\
&= \frac{2}{L}\left[\int_0^{L/2} x\sin\left(\frac{n\pi}{L}x\right)dx + \int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi}{L}x\right)dx\right] \\
&= \frac{2}{L}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2} + \frac{L}{n\pi}\int_0^{L/2}\cos\left(\frac{n\pi x}{L}\right)dx\right. \\
&\qquad\qquad \left. - \frac{L(L-x)}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L} - \frac{L}{n\pi}\int_{L/2}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right] \\
&= \frac{2}{L}\left[-\frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2}\right. \\
&\qquad\qquad \left. + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\Big|_{L/2}^{L}\right] \\
&= \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right) + \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi}{2}\right)\right] = \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

$$\Downarrow$$

$$y(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)$$

Since

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{k+1} & \text{if } n = 2k-1 \end{cases}$$

we have

$$A_{2n} = 0$$

$$A_{2n-1} = \frac{4L}{(2n-1)^2\pi^2}(-1)^{n+1}$$

for $n = 1, 2, 3, \cdots\cdots$ and hence the solution can also be re-written as

$$y(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left[\frac{(2n-1)\pi}{L}x\right]\cos\left[\frac{(2n-1)\pi a}{L}t\right]$$

The number $\lambda = n^2\pi^2/L^2$ *are eigenvalues, and the functions* $\sin(n\pi x/L)$*, or nonzero multiple thereof, are eigenfunctions.*

The eigenvalues carry information about the frequencies of the individual sine waves which are superimposed to form the final solution.

The Wave Equation with Zero Initial Displacement

Now let us consider the case in which the string is released from its horizontal stretched position (zero initial displacement) but with a nonzero initial velocity. The boundary value modelling this phenomenon is

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (0 < x < L,\ \ t > 0)$$

$$y(0,t) = y(L,t) = 0 \qquad (t > 0)$$

$$y(x,0) = 0 \qquad (0 < x < L)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \qquad (0 < x < L)$$

Up to a point the analysis is the same as in the preceding problem. Set $y(x,t) = X(x)T(t)$ to get

$$X'' + \lambda X = 0$$

$$X(0) = X(L) = 0$$

and

$$T'' + \lambda a^2 T = 0$$

The problem for X is the same as that encountered previously, so the eigenvalues are

$$\lambda = \frac{n^2\pi^2}{L^2}$$

for $n = 1, 2, 3, \cdots$ and corresponding eigenfunctions are

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Now, however, we come to the difference between this problem and the preceding one. Here $y(x,0) = X(x)T(0)$, so $T(0) = 0$. Since we know the values of λ, the problem for T can be re-written as

$$T'' + \frac{n^2\pi^2 a^2}{L^2}T = 0, \qquad T(0) = 0$$

The general solution of this differential equation for T is

$$T_n(t) = c_n \cos\left(\frac{n\pi at}{L}\right) + d_n \sin\left(\frac{n\pi at}{L}\right)$$

Since $T(0) = 0 = c_n$,

$$T_n(t) = d_n \sin\left(\frac{n\pi at}{L}\right)$$

For each positive integer n, we now have

$$y_n(x,t) = d_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi at}{L}\right)$$

Each of these functions satisfies the wave equation, the boundary conditions, and $y_n(x,0) = 0$.

To satisfy the initial velocity condition, write a superposition

$$y(x,t) = \sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi at}{L}\right)$$

$$\Downarrow$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi a}{L} d_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi at}{L}\right)$$

So we must choose the coefficients to satisfy

$$\frac{\partial y}{\partial t}(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} d_n \sin\left(\frac{n\pi x}{L}\right)$$

Note that this is a Fourier sin expansion of $g(x)$ on $[0, L]$. By choosing

$$\frac{n\pi a}{L} d_n = \frac{2}{L} \int_0^L g(\xi) \sin\left(\frac{n\pi\xi}{L}\right) d\xi$$

or

$$d_n = \frac{2}{n\pi a} \int_0^L g(\xi) \sin\left(\frac{n\pi\xi}{L}\right) d\xi$$

we obtain the solution

$$y(x,t) = \frac{2}{\pi a} \sum_{n=1}^{\infty} \frac{1}{n} \left[\int_0^L g(\xi) \sin\left(\frac{n\pi\xi}{L}\right) d\xi\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi at}{L}\right)$$

Example

We consider the same wave equation as in the preceding one but with zero initial displacement and a initial velocity

$$g(x) = \begin{cases} x & 0 \le x \le L/4 \\ 0 & L/4 < x \le L \end{cases}$$

Compute

$$\int_0^L g(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi = \int_0^{L/4} \xi\sin\left(\frac{n\pi\xi}{L}\right)d\xi$$

$$= \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) - \frac{L^2}{4n\pi}\cos\left(\frac{n\pi}{4}\right)$$

The solution for the above problem is then given by

$$y(x,t) = \frac{2L^2}{\pi^2 a}\sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}\sin\left(\frac{n\pi}{4}\right) - \frac{1}{4n}\cos\left(\frac{n\pi}{4}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi at}{L}\right)$$

The Wave Equation with Initial Displacement and Velocity

Consider the boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \quad (0 < x < L, \;\; t > 0)$$

$$y(0,t) = y(L,t) = 0 \quad (t > 0)$$

$$y(x,0) = f(x) \quad (0 < x < L)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \quad (0 < x < L)$$

This models a vibrating string in which we have possibly nonzero initial displacement and velocity function.

Instead of solving this problem directly, we can solve two problem separately, namely,

Problem 1: Zero velocity problem

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \quad (0 < x < L, \;\; t > 0)$$

$$y(0,t) = y(L,t) = 0 \quad (t > 0)$$

$$y(x,0) = f(x) \quad (0 < x < L)$$

$$\frac{\partial y}{\partial t}(x,0) = 0 \quad (0 < x < L)$$

Problem 2: Zero displacement problem

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \quad (0 < x < L, \;\; t > 0)$$

$$y(0,t) = y(L,t) = 0 \quad (t > 0)$$

$$y(x,0) = 0 \quad (0 < x < L)$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \quad (0 < x < L)$$

We have the following theorem.

Theorem 12.1. *Let y_1 be the solution of Problem 1 and y_2 be the solution of Problem 2. Then $y = y_1 + y_2$ is the solution of the entire problem with initial displacement and velocity.*

We will not give any proof for this theorem. Instead we will illustrate this in the following example.

Example: Consider the wave equation with usual boundary and with

$$f(x) = \begin{cases} x & 0 \le x \le L/2 \\ \\ L - x & L/2 \le x \le L \end{cases}$$

and initial velocity

$$g(x) = \begin{cases} x & 0 \le x \le L/4 \\ \\ 0 & L/4 < x \le L \end{cases}$$

Then it follows from the above theorem and the previous examples, the solution to the above problem is given by

$$y(x,t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a t}{L}\right)$$

$$+ \frac{2L^2}{\pi^2 a} \sum_{n=1}^{\infty} \left[\frac{1}{n^2 \pi} \sin\left(\frac{n\pi}{4}\right) - \frac{1}{4n} \cos\left(\frac{n\pi}{4}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi a t}{L}\right)$$

13. Fourier Series Solution of the Heat Equation

In what follows, we will apply the **separation of variables** method to the typical boundary value problems associated with the heat equation.

End of the Bar Kept at Zero Temperature

We want to determine the temperature distribution $u(x,t)$ in a thin homogeneous bar of length L, given the initial temperature distribution throughout the bar at time $t = 0$, if the ends are maintained at zero temperature for all time.

The boundary value problem is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, \;\; t > 0)$$

$$u(0,t) = u(L,t) = 0 \qquad (t > 0)$$

$$u(x,0) = f(x) \qquad (0 < x < L)$$

We will apply the separation of variables method and seek a solution

$$u(x,t) = X(x)T(t)$$

Substitute this into the heat equation

$$XT' = a^2 X'' T$$

$$\Downarrow$$

$$\frac{T'}{a^2 T} = \frac{X''}{X}$$

Since x and t are independent variables, both sides of this equation must be equal to the same constant.

For some λ,

$$\frac{T'}{a^2 T} = \frac{X''}{X} = -\lambda$$

Then we have

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda a^2 T = 0$$

Note that the condition that $u(0,t) = 0$ implies that $X(0)T(t) = 0$ for $t > 0$, and hence that $X(0) = 0$, assuming that $T(t)$ is not identically zero.

Similarly, $u(L,t) = 0$ implies that $X(L)T(t) = 0$ and hence $X(L) = 0$.

$$\Downarrow$$

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda a^2 T = 0 \quad X(0) = X(L) = 0$$

Unlike the wave equation, this equation for T is of first order, with no boundary condition.

The boundary value problem for X, however, is identical with that encountered with the wave equation, so we have eigenvalues

$$\lambda_n = \frac{n^2 \pi^2}{L^2}$$

and the corresponding eigenfunctions

$$X_n(x) = d_n \sin\left(\frac{n\pi x}{L}\right)$$

for $n = 1, 2, 3, \cdots$ The constant d_n will be determined later. With the values we have for λ, the differential equation for T is

$$T' + \frac{a^2 n^2 \pi^2}{L^2} T = 0$$

with general solution

$$T_n(t) = \alpha_n e^{-n^2\pi^2 a^2 t/L^2}$$

for $n = 1, 2, 3, \cdots$

Now let

$$u_n(x,t) = X_n(x)T_n(t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 a^2 t/L^2}, \qquad A_n = d_n \alpha_n$$

Each $u_n(x,t)$ satisfies the heat equation and both boundary conditions:

$$u(0,t) = u(L,t) = 0$$

There remains to satisfy the initial condition

$$u(x,0) = f(x) \qquad (0 < x < L)$$

If we can choose some positive integer n and a constant A_n such that $u_n(x,0) = f(x)$, we will have the solution.

For example, if $f(x) = 4\sin(3\pi x/L)$, we can choose $n = 3$ and $A_n = 4$ to obtain the solution

$$u(x,t) = u_3(x,t) = 4\sin\left(\frac{3\pi x}{L}\right) e^{-9\pi^2 a^2 t/L^2}$$

Usually, we cannot choose any n and A_n such that $u_n(x,0) = f(x)$, and we should attempt an infinite superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 a^2 t/L^2}$$

The condition $u(x,0) = f(x)$ requires that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

This is the Fourier series expansion of f on $[0, L]$. Hence, choose the A_n's as the Fourier coefficients:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(\xi)\sin\left(\frac{n\pi \xi}{L}\right) d\xi$$

$$\Downarrow$$

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right) d\xi\right]\sin\left(\frac{n\pi x}{L}\right) e^{-n^2\pi^2 a^2 t/L^2}$$

Example

Suppose that the bar has length $L = \pi$ and the initial temperature function is $f(x) = 2$ for $0 < x < \pi$.

Compute

$$\int_0^\pi 2\sin(n\xi)d\xi = -\frac{2}{n}\cos(n\xi)\Big|_0^\pi = -\frac{2}{n}[\cos(n\pi) - 1] = \frac{2}{n}[1 - (-1)^n]$$

The proposed solution is

$$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2}{n}[1-(-1)^n]\sin(nx)e^{-n^2a^2t}$$

$$= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin[(2n-1)x]e^{-(2n-1)^2a^2t}$$

It can be shown that this solution is also unique, as might be expected from the physical setting of the problem.

For solutions to the heat equations with (1) Temperature in a Bar with Insulated Ends and (2) Temperature Distribution in a Bar with Radiating End, please read through pages 786 to 792 of O'Neil's text.

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