# EE 2402 Engineering Mathematics III

Solutions to Tutorial 1

1. Convergence Test of Series:

   (a) Show that the series

$$\ln(a+h) = \ln a + \frac{h}{a} - \frac{h^2}{2a^2} + \frac{h^3}{3a^3} - \cdots, a > 0,$$

converges for $|h| < a$.

**Proof:** The given series can be rewritten as

$$\ln(a+h) = \ln a + \sum_{n=1}^{\infty} A_n h^n$$

with

$$A_n = \frac{(-1)^{n+1}}{na^n}$$

Compute

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{na^n}{(n+1)a^{n+1}} = \frac{1}{a}.$$

The series converges absolutely for any $|h| < a$.

   (b) Find the open interval of absolute convergence for the power series $\sum_{n=1}^{\infty} u_n(x)$ for

$$u_n(x) = \frac{(-4)^n}{n(n+1)}(x+2)^{2n}.$$

**Solution:** For $u_n(x) = \dfrac{(-4)^n}{n(n+1)}(x+2)^{2n}$, we rewrite it as

$$u_n(z) = A_n z^n,$$

with $A_n = (-4)^n/\{n(n+1)\}$ and $z = (x+2)^2$. Compute

$$\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{4^{n+1}n(n+1)}{4^n(n+1)(n+2)} = 4.$$

The series converges absolutely for any $|z| < 1/4$ or $(x+2)^2 < 1/4$ or $-5/2 < x < -3/2$.

2. Use the recurrence equations of Bessel functions in the lecture notes and the fact that for any integer $n$, $J_{-n}(x) = (-1)^n J_n(x)$ for to show that:

   (a) $J_0'(x) = -J_1(x)$

   (b) $\int xJ_0(x)dx = xJ_1(x)$

**Solution:** Given the recurrence equations

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

and

$$nJ_n(x) = \frac{x}{2}[J_{n-1}(x) + J_{n+1}(x)]$$

we have

(a)

$$J_0'(x) = \frac{1}{2}[J_{-1}(x) - J_1(x)]$$

Also, from the relation

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{for integer value of } n$$

we have

$$J_{-1}(x) = -J_1(x)$$

$$\Downarrow$$

$$J_0'(x) = \frac{1}{2}[-J_1(x) - J_1(x)] = -J_1(x)$$

(b) From the relation of Bessel functions (see lecture notes)

$$\frac{d}{dt}[t^p J_p(t)] = t^p J_{p-1}(t)$$

$$\Downarrow$$

$$\int t^p J_{p-1}(t)dt = t^p J_p(t)$$

Let $t = x$ and $p = 1$. We obtain

$$\int xJ_0(x)dx = xJ_1(x)$$

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3. The generating function for Bessel function is

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

By differentiating both sides of the equation with respect to $t$ and equating coefficients of like powers of $t$, show that

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

**Solution:** First note that

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

Differentiating both sides with respect to $t$, we have

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} \left[\frac{x}{2} + \frac{x}{2t^2}\right] = \sum_{n=-\infty}^{\infty} nt^{n-1} J_n(x)$$

$$\Downarrow$$

$$\sum_{n=-\infty}^{\infty} t^n J_n(x) \left[\frac{x}{2} + \frac{x}{2t^2}\right] = \sum_{n=-\infty}^{\infty} nt^{n-1} J_n(x)$$

$$\Downarrow$$

$$\frac{1}{2}x \sum_{n=-\infty}^{\infty} t^n J_n(x) + \frac{1}{2}xt^{-2} \sum_{n=-\infty}^{\infty} t^n J_n(x) = \sum_{n=-\infty}^{\infty} nt^{n-1} J_n(x)$$

Collecting coefficients of $t^{n-1}$, we have

$$\frac{1}{2}xJ_{n-1}(x) + \frac{1}{2}xJ_{n+1}(x) = nJ_n(x)$$

$$\Downarrow$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

4. Use the fact that $J_v$ is a solution of Bessel's equation of order $v$ to show that $x^a J_v(bx^c)$ is a solution of the equation

$$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - v^2c^2}{x^2}\right) y = 0$$

**Solution:** Let

$$y = x^a J_v(bx^c)$$
$$\Downarrow$$
$$y' = ax^{a-1} J_v(bx^c) + x^a bcx^{c-1} J_v'(bx^c)$$
$$\Downarrow$$

$$\begin{aligned} y'' &= a(a-1)x^{a-2} J_v(bx^c) + ax^{a-1}bcx^{c-1} J_v'(bx^c) + bc(a+c-1)x^{a+c-2} J_v'(bx^c) \\ &\qquad + x^{a+c-1}(bc)^2 x^{c-1} J_v''(bx^c) \\ &= a(a-1)x^{a-2} J_v(bx^c) + (2abc + bc^2 - b)x^{a+c-2} J_v'(bx^c) + b^2c^2x^{a+2c-2} J_v''(bx^c) \end{aligned}$$

Substituting the above equations to

$$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - v^2c^2}{x^2}\right) y$$

we can verify that the above expression is equal to

$$c^2x^{a-2}\left\{(bx^c)^2 J_v''(bx^c) + bx^c J_v'(bx^c) + [(bx^c)^2 - v^2] J_v(bx^c)\right\}$$

or

$$c^2x^{a-2}\left\{z^2 J_v''(z) + zJ_v'(z) + (z^2 - v^2) J_v(z)\right\}$$

with $z = bx^c$. Noting that $J_v$ is a solution of the Bessel's equation of order $v$, we have

$$z^2 J_v''(z) + zJ_v'(z) + (z^2 - v^2) J_v(z) = 0$$
$$\Downarrow$$
$$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - v^2c^2}{x^2}\right) y = 0$$

Hence $y = x^a J_v(bx^c)$ is a solution of the above equation.

5. Use the fact that $[x^{-v}J_v(x)]' = -x^{-v}J_{v+1}(x)$ and

$$J_{3/2} = \sqrt{\frac{2}{\pi x}}\left[\frac{\sin(x)}{x} - \cos(x)\right]$$

to show that

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin(x) - \frac{3}{x}\cos(x)\right]$$

**Solution:** Using the fact that $[x^{-v}J_v(x)]' = -x^{-v}J_{v+1}(x)$, we have

$$\begin{aligned}
x^{-3/2}J_{5/2}(x) &= -[x^{-3/2}J_{3/2}(x)]' \\
&= -\left[x^{-2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin(x)}{x} - \cos(x)\right)\right]' \\
&= -\sqrt{\frac{2}{\pi}}[x^{-3}(\sin(x) - x\cos(x))]' \\
&= -\sqrt{\frac{2}{\pi}}[-3x^{-4}(\sin(x) - x\cos(x)) + x^{-3}(x\sin(x))] \\
&= \sqrt{\frac{2}{\pi}}x^{-2}\left[\left(\frac{3}{x^2} - 1\right)\sin(x) - \frac{3}{x}\cos(x)\right]
\end{aligned}$$

Now multiply by $x^{3/2}$ to get

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\sin(x) - \frac{3}{x}\cos(x)\right]$$

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6. It is proved in the lecture notes that $x^a J_n(bx^c)$ and $x^a Y_n(bx^c)$ are solutions of

$$y'' - \left(\frac{2a-1}{x}\right) y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - n^2c^2}{x^2}\right) y = 0$$

for constants $a$, $b$ and $c$ and any nonnegative integer $n$. Use the above fact to write the general solutions to the following differential equations:

(a) $y'' - \frac{1}{x}y' + \left(1 - \frac{3}{x^2}\right) y = 0$

(b) $y'' - \frac{3}{x}y' + \left(\frac{1}{4x} + \frac{3}{x^2}\right) y = 0$

(c) $y'' + \frac{3}{x}y' + \frac{1}{16x}y = 0$

(d) $y'' - \frac{3}{x}y' + \left(4x^2 - \frac{60}{x^2}\right) y = 0$

**Solution:**

(a) Set

$$2a - 1 = 1, \qquad 2c - 2 = 0, \qquad b^2c^2 = 1, \qquad a^2 - n^2c^2 = -3$$
$$\Downarrow$$
$$a = 1, \qquad b = 1, \qquad c = 1, \qquad n = 2$$

Then the general solution is given by

$$y(x) = c_1 x J_2(x) + c_2 x Y_2(x)$$

(b) Set

$$2a - 1 = 3, \qquad 2c - 2 = -1, \qquad b^2c^2 = \frac{1}{4}, \qquad a^2 - n^2c^2 = 3$$
$$\Downarrow$$
$$a = 2, \qquad b = 1, \qquad c = \frac{1}{2}, \qquad n = 2$$

Then the general solution is given by

$$y(x) = c_1 x^2 J_2(\sqrt{x}) + c_2 x^2 Y_2(\sqrt{x})$$

(c) Set

$$2a - 1 = -3, \qquad 2c - 2 = -1, \qquad b^2c^2 = \frac{1}{16}, \qquad a^2 - n^2c^2 = 0$$
$$\Downarrow$$
$$a = -1, \qquad b = \frac{1}{2}, \qquad c = \frac{1}{2}, \qquad n = 2$$

Then the general solution is given by

$$y(x) = c_1 x^{-1} J_2(\sqrt{x}/2) + c_2 x^{-1} Y_2(\sqrt{x}/2)$$

(d) Set

$$2a-1=3, \qquad 2c-2=2, \qquad b^2c^2=4, \qquad a^2-n^2c^2=-60$$

$$\Downarrow$$

$$a=2, \qquad b=1, \qquad c=2, \qquad n=4$$

Then the general solution is given by

$$y(x)=c_1x^2J_4(x^2)+c_2x^2Y_4(x^2)$$

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