EE 2402 Engineering Mathematics III

Solutions to Tutorial 3

1. For $n = 0, 1, 2, 3, 4, 5$ verify that $P_n(x)$ is a solution of Legendre's equation with $\alpha = n$.

Solution: Recall the Legendre's equation from your text or lecture notes

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$$

For $\alpha = n = 0$, we have $P_0(x) = 1 \Longrightarrow P_0'(x) = P_0''(x) = 0$ and

$$(1 - x^2)P_0''(x) - 2xP_0'(x) + 0 \cdot (0 + 1)P_0(x) = 0$$

Hence $P_0(x) = 1$ is the solution for Legendre's equation with $\alpha = 0$.

For $\alpha = n = 1$, we have $P_1(x) = x \Longrightarrow P_1'(x) = 1$, $P_1''(x) = 0$ and

$$(1 - x^2)P_1''(x) - 2xP_1'(x) + 1 \cdot (1 + 1)P_1(x) = -2x + 2x = 0$$

Hence $P_1(x) = x$ is the solution for Legendre's equation with $\alpha = 1$.

For $\alpha = n = 2$, we have $P_2(x) = (3x^2 - 1)/2 \Longrightarrow P_2'(x) = 3x$, $P_2''(x) = 3$ and

$$(1 - x^2)P_2''(x) - 2xP_2'(x) + 2 \cdot (2 + 1)P_2(x) = 3(1 - x^2) - 6x^2 + 6(3x^2 - 1)/2 = 0$$

Hence $P_2(x)$ is the solution for Legendre's equation with $\alpha = 2$.

$$\vdots$$

For $\alpha = 5$, we have $P_5(x) = (63x^5 - 70x^3 + 15x)/8$, $P_5'(x) = (315x^4 - 210x^2 + 15)/8$, $P_5''(x) = (315x^3 - 105x)/2$ and

$$\begin{aligned}
&(1 - x^2)P_5''(x) - 2xP_5'(x) + 5 \cdot (5 + 1)P_5(x) \\
&\quad = (1-x^2)(315x^3-105x)/2-2x(315x^4-210x^2+15)/8+30(63x^5-70x^3+15x)/8 \\
&\quad = \frac{315}{2}x^3-\frac{315}{2}x^5-\frac{105}{2}x+\frac{105}{2}x^3-\frac{315}{4}x^5+\frac{105}{2}x^3-\frac{15}{4}x+\frac{945}{4}x^5-\frac{525}{2}x^3+\frac{225}{4}x \\
&\quad = 0
\end{aligned}$$

Hence $P_5(x)$ is the solution for Legendre's equation with $\alpha = 5$. ◇◇◇

2. Expand each of the following in a series of Legendre's polynomials:

   (a) $1 + 2x - x^2$

   (b) $2x + x^2 - 5x^3$

   (c) $2 - x^2 + 4x^4$

Solution: Using a bit of matrix notation and the formulas given in the text or your lecture notes for $P_0(x)$, $P_1(x)$, $\cdots$, $P_4(x)$, we can write

$$\begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1/2 & 0 & 3/2 & 0 & 0 \\ 0 & -3/2 & 0 & 5/2 & 0 \\ 3/8 & 0 & -30/8 & 0 & 35/8 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

Inverting the above matrix gives

$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 & 0 \\ 0 & 3/5 & 0 & 2/5 & 0 \\ 1/5 & 0 & 4/7 & 0 & 8/35 \end{bmatrix} \begin{bmatrix} P_0(x) \\ P_1(x) \\ P_2(x) \\ P_3(x) \\ P_4(x) \end{bmatrix}$$

which easily allows us to express powers of x through x^4 in terms of Legendre polynomials. Then

(a) $1 + 2x - x^2 = \frac{2}{3}P_0(x) + 2P_1(x) - \frac{2}{3}P_2(x).$

(b) $2x + x^2 - 5x^3 = \frac{1}{3}P_0(x) - P_1(x) + \frac{2}{3}P_2(x) - 2P_3(x)$

(c) $2 - x^2 + 4x^4 = \frac{37}{15}P_0(x) + \frac{34}{21}P_2(x) + \frac{32}{35}P_4(x)$

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3. Let n be a nonnegative integer. Use the fact that $P_n(x)$ is one solution of Legendre's equation with $\alpha = n$ to obtain a second, linearly independent solution

$$Q_n(x) = P_n(x) \int \frac{1}{P_n(x)^2(1-x^2)} dx$$

Hint: Let $Q_n(x) = P_n(x)z(x)$ and then substitute it to the Legendre's equation to find $z(x)$.

Solution: Let $Q_n(x) = P_n(x)z(x)$. We have

$$Q_n'(x) = P_n'(x)z(x) + P_n(x)z'(x)$$

and

$$Q_n''(x) = P_n''(x)z(x) + 2P_n'(x)z'(x) + P_n(x)z''(x)$$

Substituting into Legendre's equation, we have

$$\begin{aligned}
&(1-x^2)[P_n''z + 2P_n'z' + P_nz''] - 2x[P_n'z + P_nz'] + n(n+1)P_nz \\
&\quad = z[(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n] + z''(1-x^2)P_n + z'[2(1-x^2)]P_n' - 2xP_n] \\
&\quad = 0 + z''(1-x^2)P_n + z'[2(1-x^2)]P_n' - 2xP_n]
\end{aligned}$$

Thus, we see that $Q_n(x)$ is a solution of Legendre's equation if and only if we choose $z(x)$ such that

$$z''(1-x^2)P_n + z'[2(1-x^2)]P_n' - 2xP_n] = 0$$

$$\Downarrow$$

$$\frac{z''}{z'} + \frac{2P_n'}{P_n} - \frac{2x}{1-x^2} = 0$$

Integrating this equation, we have

$$\ln|z'| + \ln|1-x^2| + 2\ln|P_n| = c$$

$$\Downarrow$$

$$\ln\left|z'(1-x^2)(P_n)^2\right| = c$$

$$\Downarrow$$

$$z'(1-x^2)(P_n)^2 = K$$

$$\Downarrow$$

$$z'(x) = \frac{K}{(1-x^2)[P_n(x)]^2}$$

Integrate again the above equation to get

$$z(x) = K \int \frac{1}{(1-x^2)[P_n(x)]^2} dx$$

$$\Downarrow$$

$$Q_n(x) = P_n(x) \int \frac{1}{(1-x^2)[P_n(x)]^2} dx$$

which is a second linearly independent solution. Note that we have dropped K in $Q_n(x)$ as it plays no role at all. ◇◇◇

4. Use the result in Problem 3 to obtain

$$Q_0(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$
$$Q_1(x) = \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) - 1$$
$$Q_2(x) = \frac{1}{4}(3x^2-1) \ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x$$

Solution: From Problem 3 we get

$$Q_0(x) = \int \frac{1}{1-x^2} dx = \frac{1}{2} \int \left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx$$
$$= \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right)$$

for $-1 < x < 1$. Similarly,

$$Q_1(x) = x \int \frac{1}{x^2(1-x^2)} dx$$
$$= x \int \left[\frac{1}{x^2} + \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)\right] dx$$
$$= -1 + \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right)$$

$$\begin{aligned} Q_2(x) &= 2(3x^2-1) \int \frac{1}{(3x^2-1)^2(1-x^2)} dx \\ &= \frac{1}{4}(3x^2-1) \int \left[\frac{1}{x+1} - \frac{1}{x-1} + \frac{1}{(x+\frac{1}{\sqrt{3}})^2} + \frac{1}{(x-\frac{1}{\sqrt{3}})^2}\right] dx \\ &= \frac{1}{4}(3x^2-1) \ln\left(\frac{1+x}{1-x}\right) - \frac{3}{2}x \end{aligned}$$

for $-1 < x < 1$. ◇◇◇

5. Show the following properties of Legendre polynomials:

   (a) $P_n(-x) = (-1)^n P_n(x)$ for $-1 \le x \le 1$ and $n = 0, 1, 2, \cdots$

   (b) For any integer $n > 0$,
   $$\int_{-1}^{1} P_n(x)dx = 0$$
   Hint: $P_n(x) = P_n(x)P_0(x)$.

Solution:

(a) From the lecture notes or the text, we have

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}$$

and also note that

$$(-x)^{n-2k} = (-1)^{n-2k} x^{n-2k} = (-1)^n x^{n-2k}$$

$$\Downarrow$$

$$P_n(-x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} (-x)^{n-2k}$$

$$\Downarrow$$

$$P_n(-x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} (-1)^n x^{n-2k} = (-1)^n P_n(x)$$

(b)

$$\int_{-1}^{1} P_n(x)dx = \int_{-1}^{1} P_n(x)P_0(x) = 0$$

since $P_n(x)$ and $P_0(x) = 1$ are orthogonal on $[-1, 1]$ for each $n \ge 1$.

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